SILENT SOURCES ON A SURFACE FOR THE HELMHOLTZ EQUATION AND DECOMPOSITION OF L^2 VECTOR FIELDS

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Abstract. We study an inverse source problem with right hand side in divergence form for the Helmholtz equation, whose underlying model can be related to weak scattering from thin interfaces. This inverse problem is not uniquely solvable, as the forward operator has infinite-dimensional kernel. We present a decomposition of (not necessarily tangent) vector fields of L^2-class on a closed Lipschitz surface in $\mathbb{R}^3$, which allows one to discuss an *ansatz* for the solution and constraints that restore uniqueness. This work can be seen as a generalization of references [4, 6] dealing with the Laplace equation, but in the Helmholtz case new ties arise between the observations from each side of the surface. Our proof is based on properties of the Calderón projector on the boundary of Lipschitz domains, that we establish in a $H^{-1} \times L^2$ setting.

1. Introduction

Inverse source problems are classical inverse problems that relate to numerous applications, including medical imaging, ultrasound imaging, microwave imaging, or multimodal imaging techniques such as photoacoustics [17]. This work is concerned with source terms in divergence form which arise naturally, for example when modelling anisotropy in the medium response or when a static electromagnetic setting is used like in Electro-Encephalography. The corresponding inverse problems are extremely ill-posed, since the forward operator is not even injective, and thus the solution is subject to a fundamental uncertainty that can only be resolved upon making additional assumptions. The aim of this paper is to contribute to the analysis thereof in the case of the Helmholtz equation, by bringing out the structure of this uncertainty in the situation where the source is supported on a surface.

Specifically, the model problem we are interested in is governed by an equation of the form

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$$\Delta u + k^2 u = \nabla \cdot \boldsymbol{M} \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where u meets a Sommerfeld radiation condition at infinity. The left hand side of (1.1) is the Helmholtz operator with wave number k, while the right-hand side is a source term in divergence form where $\boldsymbol{M}$ is some distribution supported on a known Lipschitz surface $\Gamma \subset \mathbb{R}^3$, which is the boundary of a bounded domain $\Omega \subset \mathbb{R}^3$. Equation (1.1) can be viewed as an approximate model for scattering from thin interfaces, see Remark 2, and references [11, 7]).

A typical inverse problem associated with (1.1) is to recover $\boldsymbol{M}$ from knowledge of the field ∇u outside the surface. If $\boldsymbol{M}$ is such that the field vanishes inside (or outside) Ω, it is said to be *silent* inside (or outside). The existence of non-trivial silent $\boldsymbol{M}$ implies non-injectivity of the forward operator, and is one of the big issues facing such inverse problems.

When Γ is a compact, connected Lipschitz surface and $k = 0$, so that the left-hand side of (1.1) reduces to the ordinary Laplacian, a direct sum decomposition of $\mathbb{R}^3$-valued vector fields with components in $L^2(\Gamma)$ as an interior silent component, an exterior silent component, and a tangent divergence-free term (which is silent on both sides) was obtained in [4]; see also [6] for the case of the plane. In the present paper, we generalize such a decomposition to non-zero k and show that a fourth, finite-dimensional summand is generally required. The description of the summands allows one to structure the solutions of the inverse problem, and to specify how much information can be recovered from given data.

Our approach is different from [4] and relies heavily on properties of Calderón projectors [21], moreover it is connected with the data completion algorithm proposed in [3]. We also make intensive use of properties of singular integrals on Lipschitz surfaces expounded in [16], to derive the necessary material to handle low

(L^2) regularity. While it is possible, and in fact somewhat simpler to derive corresponding results for vector fields $\boldsymbol{M}$ whose tangential and normal components belong to $H^{1/2}(\Gamma)$, the authors feel that the L^2 theory is more natural because the membership $\boldsymbol{M} \in (L^2(\Gamma))^3$ can be defined independently of the normal frame. In contrast, the dependence of the latter on the embedding $\Gamma \to \mathbb{R}^3$ makes it difficult to intrinsically describe vector fields whose normal and tangential components lie in $H^{1/2}(\Gamma)$. And even more importantly perhaps, the L^2-framework is better suited for numerical implementations, as convergent discretization in $W^{1/2}$ is hard to handle. We work in $\mathbb{R}^3$ throughout, even though generalizing to $\mathbb{R}^n$ is straightforward.

The paper is organized as follows. In Section 2, we set up notation and conventions used for function spaces and operators in Euclidean space and on a surface. Section 3 is devoted to the statement of the problem and the characterization of silent sources in terms of Calderón projectors. The main result of the paper, namely the decomposition of $L^2(\Gamma)^3$ functions in terms of silent sources, is stated and proven in Section 4. Finally, this decomposition is illustrated by explicit calculations in the case of spheres. The paper concludes with a technical appendix containing a few results on surface potentials and elliptic regularity that we could not find in the literature; several of them are adaptations to the case $k > 0$ of results from [16].

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2. Preliminaries and Notation

If V is a topological vector space over $\mathbb{R}$ or $\mathbb{C}$ we denote by V^* its dual and we write $\mathcal{G}^* : W^* \to V^*$ for the adjoint of an operator $\mathcal{G} : V \to W$; also, $\operatorname{Ker}\mathcal{G}$ denote its kernel and $\operatorname{Im}\mathcal{G}$ its image. For $v \in V$ and $\omega \in V^*$, we let $\langle \omega, v\rangle$ indicate the duality product. If V is equipped with a conjugation ($v \mapsto \overline{v}$), we define a sesquilinear form on $V^* \times V$ by $\langle\!\langle \omega, v\rangle\!\rangle := \overline{\langle \omega, \overline{v}\rangle}$, and for a Hilbert space V we denote the inner product of $v, u \in V$ by $\langle\!\langle v, u\rangle\!\rangle_V$. Notice our convention that such products are linear on the second entry. We also identify V with V^* *via* the linear isometry $v \mapsto \langle\!\langle \overline{v}, \cdot\rangle\!\rangle_V$. When $\Omega \subset \mathbb{R}^n$ is open, we set $C^\infty(\Omega)$ to be the space of infinitely differentiable functions on Ω, and $C_c^\infty(\Omega)$ the subspace of those having compact support. We denote by $\mathcal{E}(\Omega)$ the space $C^\infty(\Omega)$ endowed with the topology of uniform convergence of all derivatives on compact sets, and by $\mathcal{D}(\Omega)$ the space $C_c^\infty(\Omega)$ equiped with the inductive topology of subspaces with support in a fixed compact set [25, Chapter I, Section 2]. Then, $\mathcal{D}^*(\Omega)$ is the space of distributions on Ω. Given $\omega \in \mathcal{D}^*(\mathbb{R}^n)$ we let $\operatorname{supp}(\omega)$ denote its support, and we write $\partial_j\omega$ for its (distributional) partial derivative with respect to the j-th coordinate in $\mathbb{R}^n$.

For $p \geq 1$ and Q a Borel set in $\mathbb{R}^3$ with ρ a positive Borel measure on Q, we let $L^p(Q,\rho)$ denote the familiar Lebesgue space of p-summable functions (essentially bounded if $p = \infty$) on Q. When ρ is Lebesgue measure, we simply write $L^p(Q)$. For $E \subset \mathbb{R}^n$, a function $f : E \to \mathbb{R}^m$ is Lipschitz if $|f(x) - f(y)| \leq k|x - y|$ for $x, y \in E$, and the smallest constant k for which this holds is the *Lipschitz constant of* f, denoted as k_f. We write $\operatorname{Lip}(E)$ for the space of Lipschitz functions E endowed with the norm $\|f\|_{L^\infty(E)} + k_f$. Such a function extends to a Lipschitz function on the whole of $\mathbb{R}^3$ [1, Theorem 7.2], and clearly the extension can be chosen to have compact support if E is bounded.

For $\Omega \subset \mathbb{R}^3$ an open set and $s \in \mathbb{R}$, let $H^s(\Omega)$ denote the Bessel potential space of order s (with index 2); the latter consists of restrictions to Ω of tempered distributions T on $\mathbb{R}^3$ whose Fourier transform $\hat{T}$ is such that $(1 + |\xi|^2)^{s/2}\hat{T} \in L^2(\mathbb{R}^3)$. On $H^s(\Omega)$ one puts the norm $\|(1 + |\xi|^2)^{s/2}\hat{T}\|_{L^2(\mathbb{R}^3)}\|$, and if $\alpha \geq 0$ then $H^s(\Omega)$ is a space of functions; see [21, ch. 3]. Clearly, $H^0(\Omega) = L^2(\Omega)$ and $H^t(\Omega) \subset H^s(\Omega)$ for $s < t$ with dense inclusion. In particular, $H^s(\mathbb{R}^3)$ densely contains compactly supported Lipschitz functions for $s \leq 1$.

A Lipschitz domain in Ω is one whose boundary is locally isometric to the graph of a Lipschitz function. If Ω is Lipschitz then $H^1(\Omega)$ coincides with functions in $L^2(\Omega)$ whose distributional derivatives again lie in $L^2(\Omega)$, moreover $H^s(\Omega)$ is the real interpolation space $[L^2(\Omega), H^1(\Omega)]_s$ and $H^{-s}(\Omega) = (H_0^s(\Omega))^*$, where $H_0^s(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $H^s(\Omega)$, see [21, Theorems 3.18 & 3.30 & 3.33]. In particular, $H^s(\Omega) \subset (\operatorname{Lip}(\Omega))^*$ for $s \geq -1$ as soon as Ω is Lipschitz and bounded, and in this range a member of $H^s(\Omega)$ is completely determined by its action on compactly supported Lipschitz functions in Ω. Still in the case that Ω is bounded and Lipschitz, we also define for $s \geq 0$:

defHloc (2.1)
$$H^s_\ell(\mathbb{R}^3 \setminus \overline{\Omega}) := \{\omega \in \mathcal{D}^*(\mathbb{R}^3 \setminus \overline{\Omega}) \ : \ \omega_{|\mathbb{B}_r \setminus \overline{\Omega}} \in H^s(\mathbb{B}_r \setminus \overline{\Omega}), \text{ for each } r > 0 \text{ such that } \overline{\Omega} \subset \mathbb{B}_r\},$$

where $\mathbb{B}_r \subset \mathbb{R}^3$ denotes the open ball of radius r centered at 0. This space is denoted as $H^s_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{\Omega})$ in [21], but this conflicts with standard notation which is why we adopt a subscript ℓ. We also put for convenience $H^s_\ell(\Omega) \coloneqq H^s(\Omega)$ to streamline notation at some places. This is consistent with (2.1), in that $H^s_\ell(\Omega)$ is comprised of functions lying in $H^s(\Omega \cap B_r)$ for all r large enough.

For a compact Lipschitz surface $M \subset \mathbb{R}^3$ which is the boundary of a Lipchitz open set, we let σ indicate surface measure on M; *i.e.*, $\sigma = \mathcal{H}^2_{|M}$, the restriction to M of 2-dimensional Hausdorff measure [28, Remark 5.8.3]. We write $L^2(M)$ for $L^2(M, \sigma)$, also for $n \geq 1$ and $\phi, \tilde{\phi} \in L^2(M)^n$ we let

$$\langle \phi, \tilde{\phi} \rangle_{L^2(M)^n} \coloneqq \int_M \phi \cdot \tilde{\phi} \, \mathrm{d}\sigma \qquad \text{and} \qquad \langle\!\langle \phi, \tilde{\phi} \rangle\!\rangle_{L^2(M)^n} \coloneqq \int_M \overline{\phi} \cdot \tilde{\phi} \, \mathrm{d}\sigma,$$

where $\overline{\phi}$ denotes the complex conjugate of ϕ. For the remaining definitions, we fix a particular compact Lipschitz surface $M \subset \mathbb{R}^3$ with atlas $\{(\theta_j, U_j)\}_{j \in I}$, in such a way that $\theta_j(U_j)$ is a ball $B_j \subset \mathbb{R}^2$ for each j and, for some rigid motion R_j of $\mathbb{R}^3$, the map $\theta_j^{-1} : B_j \to \mathbb{R}^3$ is of the form $R_j \circ (I_2 \times \psi_j)$ where I_2 is the identity operator on $\mathbb{R}^2$ and $\psi_j : B_j \to \mathbb{R}$ is Lipschitz-smooth. Without loss of generality, we assume that the charts are finitely many. A point $x \in M$ such that θ_j^{-1} is differentiable at $\theta_j(x)$ for all j such that $x \in U_j$ is called *regular*. By Rademacher's theorem, σ-a.e. $x \in M$ is regular. Defined this way regular points depend on the atlas, but this is unimportant to us; see [28, Section 5.8] for a more intrinsic definition. Given a regular point $x \in M$, we let $T_x M \subset \mathbb{R}^3$ denote the tangent space of M at x. The latter is defined as the image of the derivative $D\theta_j^{-1}(\theta_j(x))$, and by the chain rule this definition is independent of j such that $x \in U_j$. For a function $f : M \longrightarrow \mathbb{C}$ and a point $x \in U_i$ such that $f \circ \theta_j^{-1}$ is differentiable at $\theta_j(x)$, we let $\nabla_{\mathrm{T}} f(x) \in T_x M$ denote the surface gradient of f at the point x. Note that if $f : M \longrightarrow \mathbb{C}$ is Lipschitz then, for σ-a.e. $x \in M$, $\nabla_{\mathrm{T}} f(x)$ is well defined. We endow $\mathrm{Lip}(M)$ with the norm $\|f\|_\infty + \|\nabla_{\mathrm{T}} f\|_\infty$; Lipschitz partitions of unity subordinated to an open cover exist as in the smooth case. The space $\mathrm{Lip}(M)$ and Lipschitz partitions of unity will allow us to quickly define Sobolev spaces of index $s \in [-1, 1]$ on M, which is all we need. For more general cases, we refer the reader to [21, 13]. Indeed, if $\mathrm{Lip}_c(U_j)$ denotes the spaces of Lipschitz functions compactly supported in U_j, we see on using partitions of unity that a member of $\mathrm{Lip}(M)^*$ is completely determined by its effect on Lipschitz functions supported on U_j for each j. In addition, there is a one-to-one correspondence between $\mathrm{Lip}_c(B_j)$ and $\mathrm{Lip}_c(U_j)$ given by $\mathrm{Lip}_c(B_j) \ni f \mapsto f \circ \theta_j \in \mathrm{Lip}_c(U_j)$. Now, letting $\tilde{g}$ denote the extension by zero to all of M of a function initially defined on a subset of M, we put for $s \in [-1, 1]$:

defHs
$$H^s(M) \coloneqq \left\{ \psi \in \mathrm{Lip}(M)^* : \forall j \in I, \text{ the map } \mathrm{Lip}_c(B_j) \ni f \mapsto \langle \psi, \widetilde{f \circ \theta_j} \rangle \text{ belongs to } H^s(B_j) \right\}. \tag{2.2}$$

Moreover, if we write $\psi^{\theta_j} : \mathrm{Lip}_c(B_j) \to \mathbb{R}$ for the map $\psi^{\theta_j}(f) \coloneqq \langle \psi, \widetilde{f \circ \theta_j} \rangle$ above, we define the convergence of a sequence $(\psi_n)_n \subset H^s(M)$ to $\psi \in H^s(M)$ as the convergence $\psi_n^{\theta_j} \to \psi^{\theta_j}$ in $H^s(B_j)$ for any $j \in I$. This convergence is independent of the atlas and the $H^s(M)$ are Hilbert spaces. Again, for $s < t$ we have that $H^t(M) \subset H^s(M)$ and $H^0(M) = L^2(M)$, furthermore $H^{-s}(M)$ identifies with $(H^s(M))^*$. Note that $\mathrm{Lip}(M)$ is dense in $H^s(M)$ for all $s \in [-1, 1]$.

We refer on several occasions to results from [16] that uses a more general definition of Sobolev spaces, discussed for example in [14]; in the present context, it reduces to the one just described.

We say that $\boldsymbol{f} \in \mathrm{Lip}(M)^3$ (resp. $L^2(M)^3$, $H^1(M)^3$...) belongs to $\mathrm{Lip}_T(M)$ (resp. $L^2_T(M)$, $H^1_T(M)$) if, for σ-a.e. $x \in M$, it has $\boldsymbol{f}(x) \in T_x M$. Now, for a $\boldsymbol{\phi} \in L^2_T(M)$, one can define by duality the surface divergence of ϕ, denoted by $\nabla_{\mathrm{T}} \cdot \boldsymbol{\phi}$; i.e. for each $f \in \mathrm{Lip}(M)$, it is required that

$$\langle \nabla_{\mathrm{T}} \cdot \boldsymbol{\phi}, f \rangle \coloneqq -\langle \boldsymbol{\phi}, \nabla_{\mathrm{T}} f \rangle_{L^2(M)^3},$$

and then it follows from the previous definitions that $\nabla_{\mathrm{T}} \cdot \boldsymbol{\phi} \in H^{-1}(M)$. We analogously define, for $\phi \in L^2(M)$, the weak tangential gradient of ϕ which we denote by $\nabla_{\mathrm{T}} \phi$. By density, we get for $\varphi \in H^1(M)$, $\boldsymbol{\phi} \in L^2_T(M)$, $\phi \in L^2(M)$ and $\boldsymbol{\varphi} \in H^1_T(M)$ that

$$\langle \nabla_{\mathrm{T}} \cdot \boldsymbol{\phi}, \varphi \rangle = -\langle \boldsymbol{\phi}, \nabla_{\mathrm{T}} \varphi \rangle_{L^2(M)^3}, \quad \text{and} \quad \langle \nabla_{\mathrm{T}} \cdot \boldsymbol{\varphi}, \phi \rangle = -\langle \boldsymbol{\varphi}, \nabla \phi \rangle_{L^2(M)^3}.$$

In this paper, we often consider a bounded Lipschitz domain Ω_+ with boundary Γ, and we let $\Omega_- \coloneqq \mathbb{R}^3 \setminus \overline{\Omega_+}$. This choice of signs, where a "-" is attached to the unbounded complement of the bounded domain (itself denoted with a "+"), is as in [16] but departs from [21]; we implicitly take this discrepancy into account when quoting results from [21]. Note that Γ is a Lipschitz surface, that needs not be connected in general. As a short hand, unless stated otherwise, we use the symbol $\pm$ to mean *both* + *and* -, and we employ the symbol $\mp$ to designate the opposite sign to $\pm$.

Using [16, Theorem 4.3.6] together with Lemma A.1 and its proof (see equation (A.1)), we get that

$$H^1(\Gamma) = \left\{\varphi \in L^2(\Gamma) : \nabla_{\mathrm{T}}\varphi \in L^2(\Gamma)^3\right\}.$$

For $\varphi, \tilde{\varphi} \in H^1(\Gamma)$, we have that $\nabla_{\mathrm{T}}\cdot\nabla_{\mathrm{T}}\varphi \in H^{-1}(\Gamma) \equiv H^1(\Gamma)^*$ and $\langle\nabla_{\mathrm{T}}\cdot\nabla_{\mathrm{T}}\varphi, \tilde{\varphi}\rangle = -\langle\nabla_{\mathrm{T}}\varphi, \nabla_{\mathrm{T}}\tilde{\varphi}\rangle_{L^2(\Gamma)^3}$. We put $\Delta_{\mathrm{T}} := \nabla_{\mathrm{T}}\cdot\nabla_{\mathrm{T}}$ which is the Laplace-Beltrami operator on Γ. We also use the Hermitian form:

$$\langle\!\langle\varphi, \tilde{\varphi}\rangle\!\rangle_{H^1(\Gamma)} := \langle\!\langle\varphi, \tilde{\varphi}\rangle\!\rangle_{L^2(\Gamma)} + \langle\!\langle\nabla_{\mathrm{T}}\varphi, \nabla_{\mathrm{T}}\tilde{\varphi}\rangle\!\rangle_{L^2(\Gamma)^3},$$

which generates the same topology on $H^1(\Gamma)$ as the one defined after (2.2) by invariance of Sobolev functions under composition with Lipchitz maps [28, Theorem 2.2.2]. We denote by $\|\cdot\|_{H^1(\Gamma)}$ the corresponding norm. Also, we denote the dual norm in $H^{-1}(\Gamma)$ by $\|\cdot\|_{H^{-1}(\Gamma)}$, and the latter arises from a Hermitian product $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{H^{-1}(\Gamma)}$. In [16], a different norm is used for this space which is equivalent to the present one.

We denote the classical trace on Γ from $\Omega_\pm$ by $\gamma^\pm : H^1(\Omega_\pm) \longrightarrow H^{1/2}(\Gamma)$, which is a bounded linear operator. If for $\phi \in H^1_\ell(\mathbb{R}^3)$ it holds that $\gamma^+\phi = \gamma^-\phi$, we simply write $\gamma\phi := \gamma^\pm\phi$.

We also use nontangential limits on Γ. That is, given $\alpha > 0$, we define a nontangential domain of approach to $x \in \Gamma$ by

$$\mathfrak{C}^\pm_\alpha(x) := \{y \in \Omega_\pm \ : \ |x-y| \leq (\alpha+1)\mathrm{dist}(y,\Gamma)\},$$

where dist indicates Euclidean distance between a point and a set. Subsequently, for ψ a measurable function on $\Omega_\pm$ and $x \in \Gamma$, we put

$$\gamma^\pm_\alpha\psi(x) := \lim_{\substack{y \to x \\ y \in \mathfrak{C}^\pm_\alpha(x)}} \psi(y)$$

whenever this limit exists. From [16, Proposition 3.3.1] it follows that x lies in $\overline{\mathfrak{C}^\pm_\alpha(x)}$ for σ-a.e. $x \in \Gamma$, hence this definition is meaningful σ-a.e. If the limit exists for every $\alpha > 0$, we say that the nontangential limit of ψ from $\Omega_\pm$ exists at x. In the case that the nontangential limit of ψ exists for σ-a.e. $x \in \Gamma$, we denote the resulting function by $\gamma^\pm\psi$ (same notation as the trace of a Sobolev function), and in case $\gamma^+\psi = \gamma^-\psi$ we likewise drop the subscript and write $\gamma\psi$.

rmk|traces

Remark 1. The apparent abuse of notation assigning the same symbol to the trace/nontangential limit is justified by the fact that for Lipschitz domains the trace coincides σ-a.e. with the nontangential limits for H^1_ℓ functions, in the case that such limits exist. One way to show this is to prove the result locally, when the boundary is a Lipschitz graph above a plane A, and apply the absolute continuity of Sobolev functions on a.e. line perpendicular to A [28, Section 2.1].

Note that the restriction mapping $\gamma : \mathcal{E}(\mathbb{R}^3) \longrightarrow \mathrm{Lip}(\Gamma)$, is continuous; we will use the symbol γ^* to denote the adjoint operator of this particular version of the trace.

sec|Stat_lay

3. Statement of the problem and layer potentials

3.1. **Statement of the problem.**

We fix throughout $k \geq 0$ and a bounded Lipschitz domain $\Omega_+ \subset \mathbb{R}^3$ with boundary Γ, surface measure σ and outward-pointing unit normal $\boldsymbol{\nu}(x)$ at σ-a.e. $x \in \Gamma$. Set $G(x) := -\frac{e^{ik|x|}}{4\pi|x|}$, which is a fundamental solution the Helmholtz equation. We use $\mathcal{G}$ to denote its potential operator, that is:

$$\begin{array}{cccc} \mathcal{G} : & \mathcal{E}^*(\mathbb{R}^n) & \longrightarrow & \mathcal{D}^*(\mathbb{R}^3) \\ & d & \mapsto & G * d. \end{array}$$

By [26, Theorem 27.6], the map $\mathcal{G}$ is continuous and injective. For $\boldsymbol{M} \in L^2(\Gamma)^3$, we write $\boldsymbol{M} = \boldsymbol{\nu}M_{\boldsymbol{\nu}} + \boldsymbol{M}_T$ with $M_{\boldsymbol{\nu}} := \boldsymbol{M}\cdot\boldsymbol{\nu}$ and $\boldsymbol{M}_T := \boldsymbol{M} - \boldsymbol{\nu}M_{\boldsymbol{\nu}}$. Clearly, $\boldsymbol{M}_T \in L^2_T(\Gamma)$, therefore one can define $\nabla_{\mathrm{T}}\cdot\boldsymbol{M}_T \in H^{-1}(\Gamma)$. We then introduce the forward operator:

$$\begin{array}{cccc} \mathcal{F} : & L^2(\Gamma)^3 & \longrightarrow & \mathcal{D}^*(\mathbb{R}^3) \\ & \boldsymbol{M} & \mapsto & \mathcal{G}[\nabla\cdot(\boldsymbol{M}\sigma)], \end{array}$$

where by $\boldsymbol{M}\sigma$ we mean the measure on $\mathbb{R}^3$ such that $\mathrm{d}(\boldsymbol{M}\sigma) = \boldsymbol{M}d\sigma = \boldsymbol{M}\mathrm{d}\mathcal{H}^1_{|\Gamma}$, and $\nabla\cdot(\boldsymbol{M}\sigma)$ is the (weak) Euclidean divergence of $\boldsymbol{M}\sigma$ in $\mathbb{R}^3$. Note that $\mathcal{F}(\boldsymbol{M}) = \nabla G * (\boldsymbol{M}\sigma)$ and thus, $\mathcal{F}(\boldsymbol{M})$ is a locally integrable function on $\mathbb{R}^3$ which is real analytic on $\mathbb{R}^3 \setminus \Gamma$. If we set $u = \mathcal{F}(\boldsymbol{M})$, then u satisfies the Helmholtz equation:

eq|helmholtz

$$\Delta u + k^2 u = \nabla \cdot (\boldsymbol{M}\sigma), \tag{3.1}$$

as well as the Sommerfeld radiation condition:

eq|RC

$$\lim_{|x|\to\infty} |x| \left(\frac{\partial}{\partial |x|} - ik \right) u(x) = 0. \tag{3.2}$$

Since $\mathcal{G}$ is injective, the kernel of $\mathcal{F}$ consists of those $\boldsymbol{M} \in L^2(\Gamma)^3$ such that $\nabla \cdot (\boldsymbol{M}\sigma) = 0$. Also, as $\mathcal{F}(M)$ is continuous off Γ and $\mathcal{F}(M) \in L^1_{loc}(\mathbb{R}^3)$, membership of M in that kernel is tantamount to $\mathcal{F}(M)$ being identically zero on $\mathbb{R}^3 \setminus \Gamma$. In other words, we have the following chain of equivalences:

equi|F=0_fun

$$\mathcal{F}(\boldsymbol{M})(x) = 0 \text{ for all } x \in \mathbb{R}^3 \setminus \Gamma \quad \Longleftrightarrow \quad \boldsymbol{M} \in \operatorname{Ker} \mathcal{F} \quad \Longleftrightarrow \quad \nabla \cdot (\boldsymbol{M}\sigma) = 0. \tag{3.3}$$

We say that $\boldsymbol{M}$ is *silent inside* (resp. *silent outside*) if $(\mathcal{F}(\boldsymbol{M}))_{|\Omega_+} = 0$ (resp. $(\mathcal{F}(\boldsymbol{M}))_{|\Omega_-} = 0$). When $\boldsymbol{M}$ is both silent inside and silent outside, we say that it is *silent everywhere* (or simply *silent*), and if it is neither silent inside nor silent outside we say that it is *silent nowhere*. The issue that we raise is to describe the vector fields in $L^2(\Gamma)^3$ that correspond to these various notions of silence. Note that we only distinguish between silence inside and outside Ω^+: we do not consider diverse qualifications of silence in a prescribed set of components of $\mathbb{R}^3 \setminus \Omega_+$ arising when Γ is not connected, as is done for $k = 0$ in [4]. While, the present approach can be adapted for that purpose, the basic features of the problem are already present in the case that we study, and the results are simpler to state.

Note that a temperate distribution u and a vector field $\boldsymbol{M} \in L^2(\Gamma)^3$ satisfy (3.1) and (3.2) if and only if $u = \mathcal{F}(\boldsymbol{M})$. Indeed, a temperate solution T to $\Delta T + k^2 T = 0$ on $\mathbb{R}^n$ has a Fourier transform $\hat{T}$ with compact support, hence $T \in H^s(\mathbb{R}^n)$ for some s; thus, if T meets (3.2) then we can appeal to [21, Theorems 7.12 & 9.6] to conclude that $T \equiv 0$.

rmk|remamodel

Remark 2. Besides inverse magnetisation or EEG problems studied for example in [6, 4, 24] that correspond to the case $k = 0$, equation (3.1) can serve as a model for scattering from thin films [11, 7]. Indeed, consider a thin layer of constant width $\epsilon \ll 1$ coating $\partial\Omega$ with some material characterised by a coefficient β, so that the total field u_ϵ generated by some source f (compactly supported outside the thin layer) satisfies

$$\nabla \cdot (1 + \beta_\epsilon) \nabla u_\epsilon + k^2 u_\epsilon = f \quad \text{in } \mathbb{R}^3$$

together with Sommerfeld radiation condition, where $\beta_\epsilon = \beta$ inside the thin layer and is zero outside. Then, formally at least,

$$u_\epsilon = u_0 + \epsilon u_1 + o(\epsilon)$$

where (the incident field) u_0 satisfies $\Delta u_0 + k^2 u_0 = f$ in $\mathbb{R}^3$ and u_1 meets (3.1) with $\boldsymbol{M} = -A\nabla u_0$, the (anisotropic) matrix field A being defined on $\partial\Omega$ by

$$A\boldsymbol{\nu} = \frac{\beta}{1+\beta}\boldsymbol{\nu} \quad \text{and} \quad A\boldsymbol{\tau} = \beta\boldsymbol{\tau}, \ \forall \boldsymbol{\tau} \text{ tangent to } \Gamma.$$

The scattered field $u_\epsilon - u_0$ can then be approximated to the first order by ϵu_1, see [11] for a rigorous justification of this type of model in the case of thin interfaces with constant width ϵ.

3.2. **Layer potentials and Green identities in Sobolev spaces.** We recall below classical tools such as layer potentials and Calderón projectors to express the solutions to the Helmholtz equation in $\Omega_\pm$. We refer to [21] for the H^1-theory, where the density of single and double layer potentials lie in $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ respectively. However, to deal with $L^2(\Gamma)$ and $H^{-1}(\Gamma)$ densities as is necessary to handle the case that $M \in L^2(\Gamma)^3$, we need to extend the domain of definition of the operators under consideration, and for this we appeal to the work in[16]. Although the results of [16] are derived for the case $k = 0$ only, we adapt them to $k \neq 0$ in Appendix A.1. Regarding references to [21], we warn the reader that the Helmholtz equation there is minus ours, hence the fundamental solution and every other quantity linear in the latter are off by a sign with respect to the present ones; we implicitly take into account this discrepancy when quting formulas from [21].

We write $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$ for the coordinates of the unit outer normal of Γ, pointing into Ω_-. For $u \in H^1_\ell(\Omega_\pm)$ such that $\Delta u \in L^2_\ell(\Omega_\pm)$, we let $\partial^\pm_{\boldsymbol{\nu}} u \in H^{-1/2}(\Gamma)$ be the interior and exterior co-normal derivatives for the Helmholtz differential operator [21, Chapter 4]). These are well-known extensions, based on the first Green formula, of the natural definition valid for $u \in H^2_\ell(\Omega_\pm)$:

$$\partial^+_{\boldsymbol{\nu}} u = \boldsymbol{\nu} \cdot \gamma^+(\nabla u) \text{ for } u \in H^2(\Omega_+) \quad \text{and} \quad \partial^-_{\boldsymbol{\nu}} u = \boldsymbol{\nu} \cdot \gamma^-(\nabla u) \text{ for } u \in H^2_\ell(\Omega_-).$$

As with the trace, if $\partial^-_{\boldsymbol{\nu}} u = \partial^+_{\boldsymbol{\nu}} u$ for $u \in H^1_\ell(\mathbb{R}^3)$ we simply write $\partial_{\boldsymbol{\nu}} u := \partial^\pm_{\boldsymbol{\nu}} u$.

We denote the single and double layer potentials associated to (3.1) by SL and DL. Recall that $SL = \mathcal{G} \circ \gamma^*$ and $DL = \mathcal{G} \circ \partial^*_{\boldsymbol{\nu}}$ and that both are continuous and injective from $\mathrm{Lip}(\Gamma)^*$ to $\mathcal{D}^*(\mathbb{R}^3)$. In particular, we have for $x \in \mathbb{R}^3 \setminus \Gamma$ and $\phi \in L^2(\Gamma)$ [21, Equations (6.16) and (6.17)] that

defpotds
$$SL\phi(x) = \int_\Gamma G(x-y)\phi(y)\mathrm{d}\sigma(y), \quad DL\phi(x) = \int_\Gamma \partial_{\boldsymbol{\nu},y} G(x-y)\phi(y)\mathrm{d}\sigma(y), \tag{3.4}$$

where $\partial_{\boldsymbol{\nu},y}$ indicates the normal derivative with respect to the variable y. It holds the mapping properties [21, Theorem 6.11]:

ge_layer_class
$$SL : H^{-1/2}(\Gamma) \longrightarrow H^1_\ell(\mathbb{R}^3) \quad \text{and} \quad DL : H^{1/2}(\Gamma) \longrightarrow H^1_\ell(\Omega_\pm). \tag{3.5}$$

rmk|reg_J
Remark 3. Note that, for any $\boldsymbol{M} \in L^2(\Gamma)^3$, we can write $\mathcal{F}(\boldsymbol{M}) = \sum_j \partial_j SL(M_j)$. Hence, in view of Lemma A.2 and the corresponding result for harmonic functions (namely, the case $k = 0$ that follows at once from [27, Theorem 3.3 (i) & Corollary 3.5 (i)]), we get that $\mathcal{F}(\boldsymbol{M}) \in L^2_{\mathrm{loc}}(\mathbb{R}^3)^3$.

Recall the three *Green Identities:* for $u, v \in H^1(\Omega_\pm)$ with $\Delta u \in L^2(\Omega_\pm)$ and for $\pm$ to mean $+$ *or* $-$, one has by [21, Theorem 4.4 (i)]:

eqs|Green
eq|Green1
$$\langle\!\langle \nabla u, \nabla v \rangle\!\rangle_{L^2(\Omega_\pm)^3} = -\langle\!\langle \Delta u, v \rangle\!\rangle_{L^2(\Omega_\pm)} \pm \langle\!\langle \partial^\pm_{\boldsymbol{\nu}} u, \gamma^\pm v \rangle\!\rangle\,; \tag{3.6a}$$

if moreover $\Delta v \in L^2(\Omega_\pm)$, then it holds in view of [21, Theorem 4.4 (iii)] that

eq|Green2
$$\left\langle\!\!\left\langle \Delta u + k^2 u, v \right\rangle\!\!\right\rangle_{L^2(\Omega_\pm)} - \left\langle\!\!\left\langle u, \Delta v + k^2 v \right\rangle\!\!\right\rangle_{L^2(\Omega_\pm)} = \mp \langle\!\langle \gamma^\pm u, \partial^\pm_{\boldsymbol{\nu}} v \rangle\!\rangle \pm \langle\!\langle \partial^\pm_{\boldsymbol{\nu}} u, \gamma^\pm v \rangle\!\rangle \tag{3.6b}$$

and, for $u \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ with $u_{|\Omega_\pm} \in H^1_\ell(\Omega_\pm)$ satisfying (3.2) as well as

$$\Delta u_{|\Omega_\pm} + k^2 u_{|\Omega_\pm} = 0 \quad \text{in} \quad \Omega_\pm,$$

we get on applying [21, Theorem 6.10] to $\Phi_\rho u$, where $\Phi_\rho \in C^\infty_c(\mathbb{R}^n)$ is 1 on B_ρ for arbitrary large ρ, that

eq|Green3
$$u = DL(\gamma^+ u - \gamma^- u) - SL(\partial^+_{\boldsymbol{\nu}} u - \partial^-_{\boldsymbol{\nu}} u). \tag{3.6c}$$

The boundary version of layer potentials are bounded linear operators, with the mapping properties

$$S : H^{s-1}(\Gamma) \longrightarrow H^s(\Gamma) \quad \text{and} \quad K : H^s(\Gamma) \longrightarrow H^s(\Gamma)$$

for $s \in \{1, 1/2, 0\}$ (these are the only cases we need)[1]. They have for $\phi \in L^2(\Gamma)$ and σ-a.e. $x \in \Gamma$ the integral representations

eq|int_rep
$$S\phi(x) = \int_{\partial D} G(x-y)\phi(y)\mathrm{d}\sigma(y), \quad K\phi(x) = \text{p.v.} \int_{\partial D} \partial_{\boldsymbol{\nu},y} G(x-y)\phi(y)\mathrm{d}\sigma(y), \tag{3.7}$$

as well as the following jump relations for $\phi \in L^2(\Gamma)$ and $\psi \in H^{-1}(\Gamma)$:

eq|jump_S_K
$$(SL\psi)_{|\Gamma} = S\psi \quad \text{and} \quad \gamma^\pm(DL\phi) = \left(\pm\frac{1}{2} Id + K\right)\phi, \tag{3.8}$$

[1]For the case $s = 1/2$ these operators are defined in [21, Chapter 7,Eq. (7.3)] by $\gamma SL\psi$ and $\gamma^+(DL\phi) + \gamma^-(DL\phi)$ (and so their "K" which they call T differs by (minus) a factor 2 from ours); there, Equation (3.7) is proven for ϕ Lipschitz while Equation (3.8) is proven for $\psi \in H^{-1/2}(\Gamma)$ and $\phi \in H^{1/2}(\Gamma)$. When $s = 1, 0$, the case $k = 0$ is treated in [16, Proposition 3.3.2, Corollary 3.6.3, Proposition 3.6.2 and Proposition 3.6.4], and adaptation to $k \neq 0$ is made through Propositions A.3, A.4, A.5 and A.6 in the Appendix.

where Id represents the identity operator, see [21, Equation (7.5)]. In (3.8), the first relation means that $S\psi$ is well defined a.e. on Γ, and it is worth pointing out that it exists in fact as an absolutely convergent integral at quasi every point of Γ; this follows from the case $k = 0$ since a superharmonic function which is not identically $+\infty$ is finite quasi-everywhere, see [2]. Also, in the particular case where $s = 1/2$, the operator S is self-adjoint [21, Eqns. (7.2) & (7.3)]. Moreover, there is a jump relation for the normal derivative of $SL\phi$ for $\phi \in H^{-1/2}(\Gamma)$ [21, Equation (7.5)], namely:

$$\partial_{\boldsymbol{\nu}}^{\pm}(SL\phi) = \left(\mp\frac{1}{2}Id + K^*\right)\phi.$$

In another connection, the following operators are well-defined and bounded for $s \in \{1, 1/2, 0\}$ [2]:

$$T := \partial_{\boldsymbol{\nu}} DL : H^s(\Gamma) \longrightarrow H^{s-1}(\Gamma).$$

rmk|grad_DL *Remark* 4. Note that [16, Proposition 3.6.2] and Proposition A.5 further say that the linear operators $\varphi \mapsto \gamma^{\pm}(\nabla DL\varphi)$, defined $H^1(\Gamma) \longrightarrow L^2(\Gamma)^3$, are bounded.

Finally, let us introduce the *Calderón projectors*: $P^{\pm} : H^s(\Gamma) \times H^{s-1}(\Gamma) \longrightarrow H^s(\Gamma) \times H^{s-1}(\Gamma)$ is defined for $s \in \{0, 1/2, 1\}$ by block-matrix multiplication as

eq|defCalderon

$$P^{\pm}(\phi, \psi) := \begin{pmatrix} \frac{1}{2}Id \pm K & \mp S \\ \pm T & \frac{1}{2}Id \mp K^* \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} \tag{3.9}$$

where, in the case $s = 0$, the operator K^* is dual to $K : H^1(\Gamma) \longrightarrow H^1(\Gamma)$. These operators are bounded by what precedes, and clearly $P^+ + P^- = Id$. When $s = 1/2$, it is known that these operators are projections, see [21, Ex. 7.6]. So, by density and continuity, we deduce they are projections in the case $s = 0$ as well. The case $s = 1$ follows by restriction of the case $s = 1/2$ to $H^1(\Gamma) \times L^2(\Gamma)$. Hereafter, we let $P_j^{\pm}(\phi, \psi)$ denote the j-th component of $P^{\pm}(\phi, \psi)$, for $j = 1, 2$.

Note that if $\phi \in L^2(\Gamma)$ then $\gamma^*\phi$ is in fact a measure, absolutely continuous with respect to σ, such that $\mathrm{d}(\gamma^*\phi) = \phi\,\mathrm{d}\sigma$. It entails in view of the dicussion before (3.4) that

$$\mathcal{F}(\boldsymbol{M}) = -DL(M_{\boldsymbol{\nu}}) + SL(\nabla_{\mathrm{T}}\cdot\boldsymbol{M}_T),$$

which justifies the following definition of a new operator:

$$\begin{array}{cccc} \widetilde{\mathcal{F}} & : & L^2(\Gamma) \times H^{-1}(\Gamma) & \longrightarrow & \mathcal{D}^*(\mathbb{R}^3) \\ & & (\phi, \psi) & \mapsto & -DL(\phi) + SL(\psi). \end{array}$$

k|reg_tilde_F *Remark* 5. Note that $u = \widetilde{\mathcal{F}}(\phi, \psi)_{|\mathbb{R}^3 \setminus \Gamma}$ belongs to $C^{\infty}(\mathbb{R}^3 \setminus \Gamma)$ and that

$$\Delta u + k^2 u = 0 \text{ on } \Omega_{\pm}.$$

Now, Γ has finitely many components, say $\Gamma_1, ..., \Gamma_l$ (see Lemma A.12), and for $j = 1, ..., l$ we let 1_{Γ_j} be the piecewise constant function on Γ with value 1 on Γ_j and 0 elsewhere. For $\psi \in H^{-1}(\Gamma)$, we define the number $c_\psi := \sum_{j=1}^{l} \langle \psi, 1_{\Gamma_j} \rangle$ and pick $\varphi_{\psi - c_\psi} \in H^1(\Gamma)$ such that $\Delta_{\mathrm{T}}\varphi_{\psi - c_\psi} = \psi - c_\psi$; this is possible by Lemma A.10. Then, we can write

$$\widetilde{\mathcal{F}}(\phi, \psi) = \widetilde{\mathcal{F}}(\phi, \Delta_{\mathrm{T}}\varphi_{\psi - c_\psi} + c_\psi) = \mathcal{F}(\phi\boldsymbol{\nu} + \nabla_{\mathrm{T}}\varphi_{\psi - c_\psi}) + SL(c_\psi),$$

and thus, by Remark 3, the image of $\widetilde{\mathcal{F}}$ is included in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$.

[2] The case $s = 1/2$ is part of [21, Theorem 7.1]. When $s = 1$, the result for $k = 0$ follows from [16, Theorem 3.2.8, Proposition 3.6.2] and equation (3.6b); Proposition A.5 then adapts [16, Proposition 3.6.2] to the case $k \neq 0$. To deal with $s = 0$, let $\mathcal{C}$ indicate the complex conjugation operator and observe from [21, Eqns. (7.3)-(7.5)] that $T = \mathcal{C} \circ T^* \circ \mathcal{C} : H^{1/2} \to H^{-1/2}$, so we can use $\mathcal{C} \circ T^*_{|H^1(\Gamma)} \circ \mathcal{C}$ to extend $T : H^0 \to H^{-1}$.

To conclude this section, we address the fact that $\partial_{\boldsymbol{\nu}}^{\pm}u$ is currently defined for those $u \in H^1(\Omega_\pm)$ such that $\Delta u \in L^2_\ell(\Omega_\pm)$ only, whereas we shall need a definition valid for any function in the image of $\widetilde{\mathcal{F}}$. To this end, we will use the following facts and the proceeding lemma.

First, by Equation (3.8), the nontangential limits $\gamma^\pm u$ of $u = \widetilde{\mathcal{F}}(\phi,\psi)$ are well-defined and belong to $L^2(\Gamma)$. In view of (3.9), they also satisfy for any $(\tilde\phi,\tilde\psi) \in L^2(\Gamma)\times H^{-1}(\Gamma)$ such that $u = \widetilde{\mathcal{F}}(\tilde\phi,\tilde\psi)$, the relation

eq|nontan

$$\gamma^\pm u = \left(\mp\frac{1}{2}Id - K\right)(\tilde\phi) + S(\tilde\psi) = \mp P_1^\pm(\tilde\phi,\tilde\psi). \tag{3.10}$$

Second, for $(\phi,\psi) \in H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$, we get from (3.9) that

im|Calderon1

$$\text{if} \quad u = \widetilde{\mathcal{F}}(\phi,\psi) \quad \text{then} \quad (\gamma^\pm u, \partial_{\boldsymbol{\nu}}^\pm u)^t = \mp P^\pm(\phi,\psi) \quad \text{and} \quad \Delta u + k^2 u = 0 \text{ on } \Omega_\pm, \tag{3.11}$$

where the superscript "t" means "transpose". Third, we get on extending $u \in H^1_\ell(\Omega_\pm)$ by zero on $\Omega_\mp$ with $\pm$ to mean $+$ *or* $-$, and using (3.6c), the implication:

im|Calderon2

$$\Delta u + k^2 u = 0 \text{ on } \Omega_\pm \quad\Longrightarrow\quad u = -\widetilde{\mathcal{F}}(\pm\gamma^\pm u, \pm\partial_{\boldsymbol{\nu}}^\pm u) \text{ and } (\gamma^\pm u, \partial_{\boldsymbol{\nu}}^\pm u)^t = P^\pm(\gamma^\pm u, \partial_{\boldsymbol{\nu}}^\pm u). \tag{3.12}$$

Finally, the following lemma holds:

lemma|InOut

Lemma 3.1. *Let $(\phi,\psi) \in L^2(\Gamma)\times H^{-1}(\Gamma)$ and $u = \widetilde{\mathcal{F}}(\phi,\psi)$. Then, for a fixed choice of sign $\pm$ holds the equivalence:*

$$u_{|\Omega_\pm} = 0 \iff P^\pm(\phi,\psi) = 0.$$

Proof. Assume first that $(\phi,\psi) \in H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$. Then, (3.11) gives us $\Delta u + k^2u = 0$ on $\Omega_\pm$ and $(\gamma^\pm u, \partial_{\boldsymbol{\nu}}^\pm u)^t = \mp P^\pm(\phi,\psi)$. If $u_{|\Omega_\pm} = 0$, then clearly $0 = (\gamma^\pm u, \partial_{\boldsymbol{\nu}}^\pm u)$ whence $P^\pm(\phi,\psi) = 0$. Conversely, suppose that $P^\pm(\phi,\psi) = 0$ so that $(\gamma^\pm u, \partial_{\boldsymbol{\nu}}^\pm u) = 0$, by (3.11). By the mapping properties (3.5) we see that $u_{|\Omega_\pm} \in H^1_\ell(\Omega_\pm)$, and by Remark 5 we know that $u \in L^2_{\text{loc}}(R^3)$. Thus, letting $\tilde u$ be the extension by zero of $u_{|\Omega_\pm}$ to $\Omega_\mp$, Implication (3.12) gives us $\tilde u = -\widetilde{\mathcal{F}}(\pm\gamma^\pm u, \pm\partial_{\boldsymbol{\nu}}^\pm u) = 0$. Therefore, it holds indeed that $u_{|\Omega_\pm} = 0$.

Next, assume that $(\phi,\psi) \in L^2(\Gamma)\times H^{-1}(\Gamma)$ and suppose that $P^\pm(\phi,\psi) = 0$, hence $P^\mp(\phi,\psi) = (\phi,\psi)$. By density, there exist a sequence, $\big((\phi_n,\psi_n)\big)_n \subset H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$ such that $(\phi_n,\psi_n) \to (\phi,\psi)$ in $L^2(\Gamma)\times H^{-1}(\Gamma)$. On the one hand, $P^\mp(\phi_n,\psi_n)$ converges to (ϕ,ψ) in $L^2(\Gamma)\times H^{-1}(\Gamma)$ by the continuity of $P^\mp$. On the other hand, as $P^\mp(\phi_n,\psi_n) \in H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$ and the equality $P^\pm P^\mp(\phi_n,\psi_n) = 0$ mechanically holds because $(P^\pm)^2 = P^\pm = P^\pm(P^\pm + P^\mp)$, we get by the first part of the proof that $\widetilde{\mathcal{F}}(P^\mp(\phi_n,\psi_n))_{|\Omega_\pm} = 0$. Noticing that $\widetilde{\mathcal{F}}(P^\mp(\cdot,\cdot))_{|\Omega_\pm}$ is continuous from $L^2(\Gamma)\times H^{-1}(\Gamma)$ into $\mathcal{D}^*(\Omega_\pm)$, we conclude that $\widetilde{\mathcal{F}}(P^\mp(\phi_n,\psi_n))_{|\Omega_\pm} \to u_{|\Omega_\pm}$ in $\mathcal{D}^*(\Omega_\pm)$ and therefore $u_{|\Omega_\pm} = 0$.

Conversely, assume that $u_{|\Omega_\pm} = 0$ and define $\left(\tilde\phi,\tilde\psi\right) \coloneqq P^\pm(\phi,\psi)$. Then, Equation (3.10) implies that

$$\tilde\phi = P_1^\pm(\phi,\psi) = \mp\gamma^\pm u = 0.$$

Besides, $P^\mp\left(\tilde\phi,\tilde\psi\right) = P^\mp P^\pm(\phi,\psi) = 0 = P^\pm P^\mp(\phi,\psi)$ and thus, by the implication already proven, we get

$$\widetilde{\mathcal{F}}\left(P^\mp(\phi,\psi)\right)_{|\Omega_\pm} = 0 \quad\text{ and }\quad 0 = \widetilde{\mathcal{F}}\left(P^\pm(\phi,\psi)\right)_{|\Omega_\mp} = \widetilde{\mathcal{F}}\left(0,\tilde\psi\right)_{|\Omega_\mp} = SL\left(\tilde\psi\right)_{|\Omega_\mp}.$$

Moreover, by the linearity of $\widetilde{\mathcal{F}}$ and the fact that $Id = P^\mp + P^\pm$, it also holds that

$$0 = u_{|\Omega_\pm} = \widetilde{\mathcal{F}}(\phi,\psi)_{|\Omega_\pm} = \widetilde{\mathcal{F}}\left(P^\mp(\phi,\psi)\right)_{|\Omega_\pm} + \widetilde{\mathcal{F}}\left(P^\pm(\phi,\psi)\right)_{|\Omega_\pm} = \widetilde{\mathcal{F}}\left(0,\tilde\psi\right)_{|\Omega_\pm} = SL\left(\tilde\psi\right)_{|\Omega_\pm}.$$

Thus $SL\left(\tilde\psi\right)_{|\Omega_+} = SL\left(\tilde\psi\right)_{|\Omega_-} = 0$, and since SL is injective from $\text{Lip}(\Gamma)^*$ to $\mathcal{D}^*(\mathbb{R}^3)$ while $SL\left(\tilde\psi\right) = \widetilde{\mathcal{F}}\left(P^\pm(\phi,\psi)\right)$ is a locally integrable function by Remark 5, it follows that $\tilde\psi = 0$ whence $P^\pm(\phi,\psi) = 0$, as desired. □

From Lemma 3.1 it is clear that, for $(\phi,\psi), (\tilde\phi,\tilde\psi) \in L^2(\Gamma)\times H^{-1}(\Gamma)$, one has

$$\widetilde{\mathcal{F}}(\phi,\psi)_{|\Omega_\pm} = \widetilde{\mathcal{F}}(\tilde\phi,\tilde\psi)_{|\Omega_\pm} \quad\text{ if and only if }\quad P^\pm(\phi,\psi) = P^\pm(\tilde\phi,\tilde\psi).$$

Now, based on (3.11), we *define* for $u = \widetilde{\mathcal{F}}(\phi,\psi)$ with $(\phi,\psi) \in L^2(\Gamma)\times H^{-1}(\Gamma)$ and $u^\pm = u_{|\Omega_\pm}$:

$$\partial_{\boldsymbol{\nu}}^\pm u = \partial_{\boldsymbol{\nu}}^\pm u^\pm \coloneqq \mp P_2^\pm(\phi,\psi) = -T(\phi) + \left(\mp\frac{1}{2}Id + K^*\right)(\psi),$$

which extends the classical definition of normal derivatives. Altogether, it holds in this case that

eq|CalderonTri

$$\mp P^{\pm}(\gamma^{\pm}u, \partial^{\pm}_{\boldsymbol{\nu}}u) = (\gamma^{\pm}u, \partial^{\pm}_{\boldsymbol{\nu}}u)^t = \mp P^{\pm}(\phi,\psi) \quad \text{and} \quad u_{|\Omega_{\pm}} = \widetilde{\mathcal{F}}(\gamma^{\pm}u, \partial^{\pm}_{\boldsymbol{\nu}}u). \tag{3.13}$$

rmk|tan_der

Remark 6. Using once more [16, Proposition 3.6.2] together with Proposition A.5 and Lemma A.9, we get for any $u = \widetilde{\mathcal{F}}(\varphi,\phi)$ with $(\varphi,\phi) \in H^1(\Gamma)\times L^2(\Gamma)$ that

$$\partial^{\pm}_{\boldsymbol{\nu}}u = \gamma^{\pm}(\nabla u)\cdot\boldsymbol{\nu}$$

and, by an argument similar to the one in Remark 1,

$$\gamma^{\pm}(\nabla u) = \partial^{\pm}_{\boldsymbol{\nu}}u\ \boldsymbol{\nu} + \nabla_{\mathrm{T}}\gamma^{\pm}u.$$

sec|main

4. Decomposition of $L^2(\Gamma)^3$

We start by introducing the spaces that we will use to decompose $L^2(\Gamma)^3$. First, let us define

$$\mathcal{M}_0 \coloneqq \{\boldsymbol{M}\in L^2(\Gamma)^3 \ :\ \boldsymbol{M} \text{ is silent everywhere}\}$$

and let $\mathcal{M}_0^{\perp}$ denote the subspace perpendicular to $\mathcal{M}_0$ in $L^2(\Gamma)^3$. Next, let us introduce the following subspaces of $\mathcal{M}_0^{\perp}$:

$$\mathcal{M}_- = \{\boldsymbol{M}\in\mathcal{M}_0^{\perp} \ :\ \boldsymbol{M} \text{ is silent outside }\},$$
$$\mathcal{M}_+ = \{\boldsymbol{M}\in\mathcal{M}_0^{\perp} \ :\ \boldsymbol{M} \text{ is silent inside }\}.$$

rmk|0+-

Remark 7. It follows from the definition that $\mathcal{M}_+\cap\mathcal{M}_- = \{0\}$, since this intersection consists of fields silent everywhere whereas both spaces belong to $\mathcal{M}_0^{\perp}$. Also, thanks to lemma 3.1, it holds

$$\mathcal{M}_0 \coloneqq \{\boldsymbol{M}\in L^2(\Gamma)^3 \ :\ \nabla_{\mathrm{T}}\cdot\boldsymbol{M}_T = 0 \text{ and } M_{\boldsymbol{\nu}} = 0\},$$
$$\mathcal{M}_{\pm} = \{\boldsymbol{M}\in\mathcal{M}_0^{\perp} \ :\ P^{\pm}(M_{\boldsymbol{\nu}}, \nabla_{\mathrm{T}}\cdot\boldsymbol{M}_T) = 0\},$$

and it follows easily from Lemma A.10 (the Helmholtz decomposition) that

$$\mathcal{M}_0^{\perp} = \{\boldsymbol{M}\in L^2(\Gamma)^3 \ :\ \boldsymbol{M}_T = \nabla_{\mathrm{T}}U_{\boldsymbol{M}_T}, \text{ for some } U_{\boldsymbol{M}_T}\in H^1(\Gamma)\}.$$

When $k\neq 0$, $\mathcal{M}_-$, $\mathcal{M}_+$ and $\mathcal{M}_0$ are not enough to decompose $L^2(\Gamma)^3$ in its entirety. That is, for $k\neq 0$ there exists a bounded Lipschitz domain Ω_+ with boundary Γ carrying $\boldsymbol{M}\in L^2(\Gamma)^3\setminus(\mathcal{M}_-\oplus\mathcal{M}_+\oplus\mathcal{M}_0)$, which is thus silent nowhere and whose potential in $\Omega_{\pm}$ is not generated by a distribution silent in $\Omega_{\mp}$; this does not happen when $k=0$ [4]. At the end of this section we will describe the space perpendicular to $(\mathcal{M}_-\oplus\mathcal{M}_+\oplus\mathcal{M}_0)$, but prior to this we shall introduce a space $\mathcal{M}_{\boldsymbol{\nu}}\subset L^2(\Gamma)^3$, whose elements are purely normal to Γ, that satisfies

$$\mathcal{M}_{\boldsymbol{\nu}}\oplus\mathcal{M}_-\oplus\mathcal{M}_+\oplus\mathcal{M}_0 = L^2(\Gamma)^3.$$

Let $\{\Gamma_j\}_{j\in J}$ be the family of connected components of Γ. The fact that Ω_+ is a bounded Lipschitz domain implies that J must be finite and each Γ_j has strictly positive and finite area (see for example Lemma A.12). We can index the connected components of Ω_- by Ω_-^j for $j\in J$, and assume that

- $J = \{1,\dots,n_\Gamma\}$, so that n_Γ is the number of connected components of Γ,
- Ω_-^1 is unbounded,
- for each $j>1$, the set Ω_-^j is bounded,
- for each $j\in J$, the set Γ_j is the boundary of Ω_-^j.

For $\Sigma\subset\Gamma$, we let 1_Σ denote the characteristic function of Σ in Γ. Also, for a vector space V and a family of vectors $\{v_\ell\}_{\ell\in L}\subset V$, we let $\langle v_\ell\rangle_{\ell\in L}$ denote the linear span of $\{v_\ell\}_{\ell\in L}$ in V. In order to study the dimension of $\mathcal{M}_{\boldsymbol{\nu}}$, we introduce the space $\mathcal{O}\coloneqq\langle 1_{\Gamma_j}\rangle_{j\in J}\subset H^1(\Gamma)$, and the spaces $\mathcal{N}_{\pm}$ defined on the lemma below:

na|char_tildeJ

Lemma 4.1. *For a fixed sign $\pm$, the following subspaces of $H^{1/2}(\Gamma)$ coincide:*

$$\mathcal{N}^1_{\pm} := \{\ \gamma^{\pm}u \ :\ u\in H^1_\ell(\Omega_{\pm}) \text{ satisfies (3.2)},\ \Delta u + k^2u = 0 \text{ on } \Omega_{\pm}, \text{ and } \partial^{\pm}_{\boldsymbol{\nu}}u = 0 \text{ on } \Gamma\ \}$$
$$\mathcal{N}^2_{\pm} := \{\ \phi\in H^{1/2}(\Gamma) \ :\ P^{\pm}(\phi,0) = (\phi,0)\ \}$$
$$\mathcal{N}^3_{\pm} := \{\ \phi\in H^{1/2}(\Gamma) \ :\ \phi\boldsymbol{\nu}\in\mathcal{M}_{\mp}\ \}.$$

We denote them by $\mathcal{N}_\pm$ and $\mathcal{N}_+ \cap \mathcal{N}_- = \{0\}$, moreover these spaces are finite-dimensional.

Proof. Remark 7 and the identity $P^+ + P^- = Id$ together imply that $\mathcal{N}_\pm^2 = \mathcal{N}_\pm^3$. Take now a $\gamma^\pm u \in \mathcal{N}_\pm^1$. By Implication (3.12), we have that $(\gamma^\pm u, 0) = (\gamma^\pm u, \partial_{\boldsymbol{\nu}}^\pm u) = P^\pm(\gamma^\pm u, \partial_{\boldsymbol{\nu}}^\pm u) = P^\pm(\gamma^\pm u, 0)$ and thus $\gamma^\pm u \in \mathcal{N}_\pm^2$. On the other hand, if $\phi \in \mathcal{N}_\pm^2$ and we let $u = -DL(\mp\phi)$, then $u_{|\Omega_\pm} \in H^1_\ell(\Omega_\pm)$ by (3.5) and it follows from Implication (3.11) that $\Delta u + k^2 u = 0$ on $\Omega_\pm$, and $(\gamma^\pm u, \partial_{\boldsymbol{\nu}}^\pm u) = \mp P^\pm(\mp\phi, 0) = (\phi, 0)$. Hence, $\phi \in \mathcal{N}_\pm^1$ and therefore, $\mathcal{N}_\pm^1 = \mathcal{N}_\pm^2$. We now see that all three definitions are equivalent.

If Ω_- is connected then, by uniqueness of the exterior Neumann problem for the Helmholtz equation when (3.2) is satisfied, we obtain that $\{0\} = \mathcal{N}_-^1 = \mathcal{N}_-$. Otherwise, for either choice of sign $\pm$, the sets $\Omega_\pm \smallsetminus \Omega_-^1$ are bounded and there exist Neumann eigenvalues, $\{\xi_j^\pm\}_{j=1}^\infty$, with $0 \le \xi_1^\pm \le \xi_2^\pm \le \cdots$, and $\xi_j^\pm \to \infty$ as $j \to \infty$, and corresponding eigenfunctions $\{u_j\}_{j=1}^\infty \subset H^1(\Omega_\pm \smallsetminus \Omega_-^1)$, satisfying

eq|Neumann
$$\begin{cases} -\Delta u_j = \xi_j^\pm u_j & \text{in } \Omega_\pm \smallsetminus \Omega_-^1 \\ \partial_{\boldsymbol{\nu}}^\pm u_j = 0 & \text{on } \Gamma, \end{cases} \tag{4.1}$$

where the $\{u_j\}_{j=1}^\infty$ are not identically zero and form a complete orthonormal system in $L^2(\Omega_\pm \smallsetminus \Omega_-^1)$; see [21, Chapter 9] (easily adapted to the case where Ω_- is not connected by a direct sum construction). That $\mathcal{N}_\pm$ is finite-dimensional comes from the fact that if $\phi \in \mathcal{N}_\pm$, then there can only be finitely many $j > 0$ such that $k^2 = \xi_j^\pm$, and of necessity ϕ is a linear combination of the corresponding $\gamma^\pm u_j$. Finally, the fact $\mathcal{N}_+ \cap \mathcal{N}_- = \{0\}$ comes the definition of $\mathcal{N}_\pm^3$ and Remark 7. □

Continuing towards the definition of the space $\mathcal{M}_{\boldsymbol{\nu}}$, fix an orthonormal basis of $\mathcal{O}$ with respect to the $L^2(\Gamma)$-metric, say $\{\omega_j\}_{j\in J}$, such that a subset of this basis is a basis of $\mathcal{O} \cap (\mathcal{N}_+ \oplus \mathcal{N}_-)$, and let

$$\tilde{J} := \{j \in J \ : \ \omega_j \notin \mathcal{N}_+ \oplus \mathcal{N}_-\};$$

we put $\tilde{n}_\Gamma$ for the cardinality of $\tilde{J}$.

For each $j \in \tilde{J}$, since J is finite and $\mathcal{N}_+ \oplus \mathcal{N}_-$ is finite-dimensional, the subspace of $L^2(\Gamma)$ defined as $V_j := \left(\mathcal{N}_+ \oplus \mathcal{N}_- \oplus \langle\omega_\ell\rangle_{\ell\neq j}^{\ell\in\tilde{J}}\right)^\perp$ is nontrivial, hence there is a nonzero $\tilde{\Lambda}_j \in V_j$ such that $\omega_j - \tilde{\Lambda}_j \in \mathcal{N}_+ \oplus \mathcal{N}_- \oplus \langle\omega_\ell\rangle_{\ell\neq j}^{\ell\in\tilde{J}}$. Then, $\Lambda_j := \tilde{\Lambda}_j / \|\tilde{\Lambda}_j\|^2_{L^2(\Gamma)}$ satisfies, $\langle\!\langle \Lambda_j, \omega_l \rangle\!\rangle_{L^2(\Gamma)} = \delta_l^j$ for $l \in J$, and $\langle\!\langle \Lambda_j, \phi \rangle\!\rangle_{L^2(\Gamma)} = 0$ for each $\phi \in \mathcal{N}_+ \oplus \mathcal{N}_-$. Therefore, by Lemma 4.1 and the Fredholm alternative (see for example [21, Chapter 9]), we can define for each $j \in \tilde{J}$ the function $u_j^+ \in H^1(\Omega_+)$ verifying

$$\begin{cases} \Delta u_j^+ + k^2 u_j^+ = 0 & \text{in } \Omega_+ \\ \partial_{\boldsymbol{\nu}}^+ u_j^+ = \Lambda_j & \text{on } \Gamma, \end{cases}$$

and, for each $j \in \tilde{J}$, we can take $u_j^- \in H^1_\ell(\Omega_-)$ verifying

$$\begin{cases} \Delta u_j^- + k^2 u_j^- = 0 & \text{in } \Omega_- \\ \partial_{\boldsymbol{\nu}}^- u_j^- = \Lambda_j & \text{on } \Gamma, \end{cases}$$

together with the Sommerfeld radiation condition, (3.2). Then, define the following functions that belong to $H^{1/2}(\Gamma)$,

$$\phi_j^+ := \gamma^- u_j^-, \quad \phi_j^- := \gamma^+ u_j^+ \quad \text{and} \quad \phi_j := \phi_j^- - \phi_j^+.$$

Finally, we let $\mathcal{M}_{\boldsymbol{\nu}} := \langle \phi_j \boldsymbol{\nu} \rangle_{j\in\tilde{J}}$. For the proofs of the results below, we will use n_Γ, $\tilde{n}_\Gamma$, $\mathcal{O}$, ω_j, ϕ_j, $\phi_j^\pm$ and Λ_j as defined above.

decomposition

Theorem 4.2. *We have the decomposition,*

eq|decomp_full
$$L^2(\Gamma)^3 = \mathcal{M}_{\boldsymbol{\nu}} \oplus \mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0, \tag{4.2}$$

where $\oplus$ denotes direct sum. Furthermore,

eq|decomp_perp
$$(\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0)^\perp = \langle \gamma\ [\nabla \mathcal{F}(\omega_j \boldsymbol{\nu})] \rangle_{j\in\tilde{J}}\,. \tag{4.3}$$

In particular, if $k = 0$, then $\mathcal{M}_{\boldsymbol{\nu}} = \{0\}$ and thus,

q|decomp_easy

$$L^2(\Gamma)^3 = \mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0. \tag{4.4}$$

On the other hand, if k^2 is not an eigenvalue for the problems in (4.1), then

$$\operatorname{codim}(\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0) = \dim(\mathcal{M}_{\boldsymbol{\nu}}) = n_\Gamma.$$

Proof. We first show that the $\phi_j \boldsymbol{\nu}$, which clearly are in $\mathcal{M}_0^\perp$, do not belong to $\mathcal{M}_- \oplus \mathcal{M}_+$ and they are indeed linearly independent. Assume for a contradiction that there exists $\boldsymbol{M}^+ \in \mathcal{M}_+$ and $\boldsymbol{M}^- \in \mathcal{M}_-$ such that $\phi_j \boldsymbol{\nu} = \boldsymbol{M}^+ + \boldsymbol{M}^-$. By equation (3.12) and the definitions of the $\phi_j^\pm$,

$$P^+(\phi_j^+, \Lambda_j) = 0 \quad \text{and} \quad P^-(\phi_j^-, \Lambda_j) = 0.$$

Then, $P^-(\phi_j, 0) = P^-(\phi_j^-, \Lambda_j) - P^-(\phi_j^+, \Lambda_j) = -(\phi_j^+, \Lambda_j)$, however, by the definitions of $\mathcal{M}_+$ and $\mathcal{M}_-$,

$$P^-(\phi_j, 0) = P^-((M_{\boldsymbol{\nu}}^-, \nabla_{\mathrm{T}} \cdot \boldsymbol{M}_T^-) + (M_{\boldsymbol{\nu}}^+, \nabla_{\mathrm{T}} \cdot \boldsymbol{M}_T^+)) = (M_{\boldsymbol{\nu}}^+, \nabla_{\mathrm{T}} \cdot \boldsymbol{M}_T^+),$$

which is not possible since $\langle\!\langle \Lambda_j, \omega_j \rangle\!\rangle_{L^2(\Gamma)} = 1$ but $\langle\!\langle \nabla_{\mathrm{T}} \cdot \boldsymbol{M}_T^+, \omega_i \rangle\!\rangle_{L^2(\Gamma)} = 0$, since the ω_i are locally constant. Therefore $\phi_j \boldsymbol{\nu} \notin \mathcal{M}_- \oplus \mathcal{M}_+$.

Now, taking a family of complex numbers $\{a_\ell\}_{\ell \in \tilde{J}}$ such that $\sum_\ell a_\ell \phi_\ell = 0$, we get that $0 = P^-(\sum_\ell a_\ell \phi_\ell, 0) = -\sum_\ell a_j(\phi_\ell^+, \Lambda_\ell)$. In particular, for any $j \in \tilde{J}$, $0 = \langle\!\langle \sum_\ell a_\ell \Lambda_\ell, \omega_j \rangle\!\rangle_{L^2(\Gamma)} = a_j$ and thus, the ϕ_j are linearly independent.

Note that, since $\mathcal{M}_{\boldsymbol{\nu}} \cap (\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0) = \{0\}$, we then have the inequality

eq|<dim

$$\tilde{n}_\Gamma = \dim \mathcal{M}_{\boldsymbol{\nu}} \leq \operatorname{codim}(\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0), \tag{4.5}$$

and thus, for equation (4.2) to hold it is only necessary to show that $\tilde{n}_\Gamma \geq \operatorname{codim}(\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0)$.

Define the following linear operators

$$\begin{array}{cccc} \pi & : & \mathcal{M}_0^\perp & \longrightarrow & L^2(\Gamma) \times H^{-1}(\Gamma) \\ & & \boldsymbol{M} & \mapsto & (M_{\boldsymbol{\nu}}, \nabla_{\mathrm{T}} \cdot \boldsymbol{M}_T) \end{array} \quad \text{and} \quad \begin{array}{cccc} \eta & : & H^{-1}(\Gamma) & \longrightarrow & \mathbb{C}^{n_\Gamma} \\ & & \psi & \mapsto & \left(\langle \psi, 1_{\Gamma_j} \rangle\right)_j . \end{array}$$

Then, by Lemma A.10, given a $(\phi, \psi) \in L^2(\Gamma) \times H^{-1}(\Gamma)$, we get the equivalence;

equi|pieta

$$(\phi, \psi) \in \operatorname{Im} \pi \quad \text{if and only if} \quad \psi \in \operatorname{Ker} \eta. \tag{4.6}$$

By Remark 7 and the fact that the projections $P^\pm$ satisfy $P^+ + P^- = Id$, it follows that

equi|MinM+-

$$\pi(\mathcal{M}_\pm) = \operatorname{Im}(P^\mp \pi) \cap \operatorname{Im} \pi. \tag{4.7}$$

Also, using again the fact that $P^+ + P^- = Id$, we obtain that, for any $\boldsymbol{M} \in \mathcal{M}_0^\perp$,

$$P^+(\pi(\boldsymbol{M})) \in \operatorname{Im} \pi \quad \text{if and only if} \quad P^-(\pi(\boldsymbol{M})) \in \operatorname{Im} \pi.$$

Hence, by equivalence (4.6), the following four inclusions are equivalent for any $\boldsymbol{M} \in \mathcal{M}_0^\perp$,

$$P_2^+(\pi(\boldsymbol{M})) \in \operatorname{Ker} \eta, \quad P^+(\pi(\boldsymbol{M})) \in \operatorname{Im} \pi, \quad P^-(\pi(\boldsymbol{M})) \in \operatorname{Im} \pi, \quad \text{and} \quad P_2^-(\pi(\boldsymbol{M})) \in \operatorname{Ker} \eta.$$

Thus, we can define

$$\begin{aligned} \Pi := \operatorname{Ker}(\eta P_2^+ \pi) &= \{\boldsymbol{M} \in \mathcal{M}_0^\perp \ : \ P^+(\pi(\boldsymbol{M})) \in \operatorname{Im} \pi\} \\ = \operatorname{Ker}(\eta P_2^- \pi) &= \{\boldsymbol{M} \in \mathcal{M}_0^\perp \ : \ P^-(\pi(\boldsymbol{M})) \in \operatorname{Im} \pi\}. \end{aligned}$$

Then, $\mathcal{M}_+ \oplus \mathcal{M}_- \subset \Pi$, hence equivalence (4.7) implies that that $\pi(\mathcal{M}_\pm) = [P^\mp \circ \pi](\Pi)$. Thus $\pi(\mathcal{M}_+ \oplus \mathcal{M}_-) = \pi(\Pi)$ and, by injectiveness of π it follows that $\mathcal{M}_+ \oplus \mathcal{M}_- = \Pi$.

For V a close subspace of $\mathcal{M}_0^\perp$, with the topology from $L^2(\Gamma)^3$, let $V^{\perp_0}$, denote the close subspace of $\mathcal{M}_0^\perp$ which is perpendicular to V and such that $V \oplus V^{\perp_0} = \mathcal{M}_0^\perp$. Now, since the $\eta P_2^\pm \pi$ are continuous on the topology of $\mathcal{M}_0^\perp$ as a subspace of $L^2(\Gamma)^3$, we get

$$(\mathcal{M}_+ \oplus \mathcal{M}_- \oplus \mathcal{M}_0)^\perp = (\mathcal{M}_+ \oplus \mathcal{M}_-)^{\perp_0} = \Pi^{\perp_0} = (\operatorname{Ker}(\eta P_2^\pm \pi))^{\perp_0} = \overline{\operatorname{Im}(\pi^*(P_2^\pm)^* \eta^*)} = \operatorname{Im}(\pi^*(P_2^\pm)^* \eta^*),$$

where the last equality comes from the fact that $\mathrm{Im}\,(\pi^*(P_2^{\pm})^*\eta^*)$ is finite dimensional and thus, closed on $\mathcal{M}_0^{\perp}$. Now, taking a $\boldsymbol{c} \in \mathbb{C}$, any $\psi \in H^{-1}(\Gamma)$, a pair $(\phi, \varphi) \in L^2(\Gamma) \times H^1(\Gamma)$ and any $\boldsymbol{M} \in \mathcal{M}_0^{\perp}$ we have,

$$\langle\!\langle \eta^* \boldsymbol{c}, \psi \rangle\!\rangle = \sum_j \overline{c_j} \langle \psi, 1_{\Gamma_j} \rangle = \left\langle \sum_j \overline{c_j} 1_{\Gamma_j},\ \psi \right\rangle = \left\langle\!\!\left\langle \sum_j c_j 1_{\Gamma_j},\ \psi \right\rangle\!\!\right\rangle,$$

eq|pi*

$$\langle\!\langle\ \pi^*(\phi,\varphi),\ \boldsymbol{M}\ \rangle\!\rangle = \langle\!\langle\ (\phi,\varphi),\ (M_{\boldsymbol{\nu}}, \nabla_{\mathrm{T}}\cdot \boldsymbol{M}_T)\ \rangle\!\rangle = \langle\!\langle \phi,\ M_{\boldsymbol{\nu}} \rangle\!\rangle - \langle\!\langle \nabla_{\mathrm{T}}\varphi,\ \boldsymbol{M}_T \rangle\!\rangle = \langle\!\langle\ \phi\boldsymbol{\nu} - \nabla_{\mathrm{T}}\varphi,\ \boldsymbol{M}\ \rangle\!\rangle_{L^2(M)^3}, \tag{4.8}$$

and, by the fact that $T^* = T$ on $H^{1/2}(\Gamma)$ and by Remark 6,

$$\begin{aligned} \mp\pi^*(P_2^{\pm})^*\eta^*(\boldsymbol{c}) &= \mp \sum_{j\in J} c_j \left[\pm (T1_{\gamma_j})\boldsymbol{\nu} - \nabla_{\mathrm{T}}\left(\frac{1}{2} 1_{\gamma_j} \mp K 1_{\gamma_j} \right) \right] \\ &= \sum_{j\in J} c_j \left[-(T1_{\gamma_j})\boldsymbol{\nu} - \nabla_{\mathrm{T}}(K1_{\gamma_j}) \right] \\ &= \sum_{j\in J} c_j \left[-(T1_{\gamma_j})\boldsymbol{\nu} - \nabla_{\mathrm{T}}\left(\pm\frac{1}{2} 1_{\gamma_j} + K1_{\gamma_j} \right) \right] = \gamma^{\pm}\left(\nabla\mathcal{F}\left(\sum_{j\in J} c_j 1_{\gamma_j} \boldsymbol{\nu} \right) \right). \end{aligned}$$

Then, the $\gamma\ [\nabla\mathcal{F}(\omega_j\boldsymbol{\nu})]$ are well defined and belong to $L^2(\Gamma)^3$ for any $j \in J$. This also shows, in light of the third definition of Lemma 4.1, that for any $j \notin \tilde{J}$ it follows that $\gamma\ [\nabla\mathcal{F}(\omega_j\boldsymbol{\nu})] = 0$. Hence, equation (4.3) is satisfied and thus,

$$\tilde{n}_\Gamma \geq \dim\left((\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0)^{\perp} \right) = \mathrm{codim}\,(\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0).$$

Therefore, by equation (4.5), it follows that $\tilde{n}_\Gamma = \mathrm{codim}\,(\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0)$, then equation (4.2) holds and the set $\{\gamma\ [\nabla\mathcal{F}(\omega_j\boldsymbol{\nu})]\}_{j\in\tilde{J}}$ consists of linearly independent functions.

Clearly, if k^2 is not an eigen-value for the problem (4.1), then $\tilde{J} = J$, and hence,

$$n_\Gamma = \tilde{n}_\Gamma = \dim(\mathcal{M}_{\boldsymbol{\nu}}) = \mathrm{codim}\,(\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0).$$

Finally, in the case $k = 0$, by noticing that

$$\mathcal{F}(1_{\Gamma_j}\boldsymbol{\nu}) = \begin{cases} -\chi_{\mathbb{R}^3 \setminus \overline{\Omega_-^j}} & \text{if } \ j = 1 \\ \chi_{\Omega_-^j} & \text{otherwise,} \end{cases}$$

it follows that $\gamma\ [\nabla\mathcal{F}(\omega_j\boldsymbol{\nu})] = 0$, for every $j \in J$, then $\tilde{n}_\Gamma = 0$, $\mathcal{M}_{\boldsymbol{\nu}} = \{0\}$, and thus equation (4.4) is satisfied. □

To finish this section we will find a characterization for the spaces $\mathcal{M}_{\pm}^{\perp}$.

Corollary 4.3. *For a fixed choice of sign* $\pm$*,*

$$(\mathcal{M}_{\pm} \oplus \mathcal{M}_0)^{\perp} = \overline{\left\{\ \gamma^{\mp}\left(\nabla\widetilde{\mathcal{F}}(\varphi,\phi) \right)\ :\ (\varphi,\phi) \in H^1(\Gamma) \times L^2(\Gamma)\ \right\}}.$$

Proof. Take η, π and Π, as defined of the proof of Theorem 4.2 and recall that $[P^{\mp} \circ \pi](\Pi) = \pi(\mathcal{M}_{\pm}) \subset \pi(\Pi)$. So, defining the bijective operator

$$\begin{array}{cccc} \pi_\Pi & : & \mathcal{M}_0^{\perp} & \longrightarrow & \pi(\Pi) \\ & & \boldsymbol{M} & \mapsto & \pi(\boldsymbol{M}), \end{array}$$

it follows that $\pi_\Pi(\mathcal{M}_{\pm}) = \mathrm{Im}\,(P^{\mp}\pi_\Pi)$ which implies that,

$$\mathcal{M}_{\pm} = \mathrm{Im}\,\left(\pi_\Pi^{-1} P^{\mp} \pi_\Pi \right)$$

For a V, subspace of Π, let $V^{\perp_\Pi}$, denote the close subspace of Π perpendicular to V and such that $V \oplus V^{\perp_\Pi} = \Pi$. Noting that the $\pi_\Pi^{-1} P^{\mp} \pi_\Pi$ are projections, we get that $\mathcal{M}_{\pm} = \mathrm{Ker}\,\left(\pi_\Pi^{-1} P^{\pm} \pi_\Pi \right)$ and thus,

eq|perp+-

$$\mathcal{M}_{\pm}^{\perp_\Pi} = \overline{\mathrm{Im}\,\left[\pi_\Pi^*(P^{\pm})^* \left(\pi_\Pi^{-1} \right)^* \right]}. \tag{4.9}$$

Now, given $\boldsymbol{M} \in \Pi = \Pi^*$, $\phi \in L^2(\Gamma)$ and $\psi \in \operatorname{Ker}\eta$, recalling the equivalence (4.6) and using Remark 7 and Lemma A.10,

$$\left\langle\!\!\left\langle \left(\pi_\Pi^{-1}\right)^* \boldsymbol{M},\ (\phi,\psi) \right\rangle\!\!\right\rangle = \langle\!\langle\ \boldsymbol{M},\ \pi_\Pi^{-1}(\phi,\psi)\ \rangle\!\rangle = \langle\!\langle\ \boldsymbol{M},\ \phi\boldsymbol{\nu} + \nabla_{\mathrm{T}}\varphi_\psi\ \rangle\!\rangle$$
$$= \langle\!\langle\ M_{\boldsymbol{\nu}},\ \phi\ \rangle\!\rangle - \langle\!\langle\ \nabla_{\mathrm{T}}\cdot U_{\boldsymbol{M}_T},\ \nabla_{\mathrm{T}}\varphi_\psi\ \rangle\!\rangle = \langle\!\langle\ M_{\boldsymbol{\nu}},\ \phi\ \rangle\!\rangle - \langle\!\langle\ U_{\boldsymbol{M}_T},\ \psi\ \rangle\!\rangle,$$

Thus, $\left(\pi_\Pi^{-1}\right)^* \boldsymbol{M} = (M_{\boldsymbol{\nu}}, -U_{\boldsymbol{M}_T}) \in L^2(\Gamma) \times H^1(\Gamma)$. Then, from equation (4.9),

eq|perp+-sub

$$\mathcal{M}_\pm^{\perp\Pi} \subset \overline{\left[\pi_\Pi^* \circ (P^\pm)^*\right](L^2(\Gamma) \times H^1(\Gamma))}. \tag{4.10}$$

Now, as $T = T^*$ on $H^{1/2}(\Gamma) \supset H^1(\Gamma)$ and $S = S^*$ on $H^{-1/2}(\Gamma) \supset L^2(\Gamma)$ we get for $(\phi,\varphi) \in L^2(\Gamma) \times H^1(\Gamma)$, that

$$(P^\pm)^*(\phi,\varphi) = \begin{pmatrix} \frac{1}{2}Id \pm K^* & \pm T \\ \mp S & \frac{1}{2}Id \mp K \end{pmatrix} \begin{pmatrix} \phi \\ \varphi \end{pmatrix}.$$

Hence, using Equation (4.8) and Remark 6, we obtain for $(\phi,\varphi) \in L^2(\Gamma) \times H^1(\Gamma)$,

$$\begin{aligned} \pm\left[\pi_\Pi^* \circ (P^\pm)^*\right](\phi,-\varphi) &= \pi_\Pi^*\left(\left(\pm\frac{1}{2}Id + K^*\right)\phi - T\varphi,\ -S\phi + \left(\mp\frac{1}{2}Id + K\right)\varphi \right) \\ &= \left(-T\varphi + \left(\pm\frac{1}{2} + K^*\right)\phi\right)\boldsymbol{\nu} \ +\ \nabla_{\mathrm{T}}\left(-\left(\mp\frac{1}{2}Id + K\right)\varphi + S\phi\right) \\ &= \partial_{\boldsymbol{\nu}}^{\mp}(-DL(\varphi) + SL(\phi)) \ +\ \nabla_{\mathrm{T}}\gamma^{\mp}(-DL(\varphi) + SL(\phi)) \\ &= \gamma^{\mp}\left(\nabla\widetilde{\mathcal{F}}(\varphi,\phi)\right). \end{aligned}$$

Then, noticing that $(\mathcal{M}_\pm \oplus \mathcal{M}_0)^\perp = \mathcal{M}_\pm^{\perp\Pi} \oplus (\Pi \oplus \mathcal{M}_0)^\perp = \mathcal{M}_\pm^{\perp\Pi} \oplus (\mathcal{M}_+ \oplus \mathcal{M}_- \oplus \mathcal{M}_0)^\perp$, and recalling Equations (4.10) and (4.3), we have the inclusion

$$(\mathcal{M}_\pm \oplus \mathcal{M}_0)^\perp \subset \overline{\left\{\ \gamma^{\mp}\left(\nabla\widetilde{\mathcal{F}}(\varphi,\phi)\right)\ :\ (\varphi,\phi) \in H^1(\Gamma) \times L^2(\Gamma)\ \right\}}.$$

Thus, to finish the proof it only remains to show the inclusion on the opposite direction for the equation above. Take any $(\varphi,\phi) \in H^1(\Gamma) \times L^2(\Gamma)$ and let $\boldsymbol{M} \coloneqq \gamma^{\mp}\left(\nabla\widetilde{\mathcal{F}}(\varphi,\phi)\right)$ and $w = \widetilde{\mathcal{F}}(\varphi,\phi)$. First note that using Remark 7 and the fact that $\boldsymbol{M}_T$ is a tangential gradient, it follows that $\boldsymbol{M} \perp \mathcal{M}_0$. Next, let $\boldsymbol{M}^\pm \in \mathcal{M}_\pm$, $w^\pm = \mathcal{F}(\boldsymbol{M}^\pm)$ and note that, by Implication 3.11 and Remark 7,

$$\pm(\gamma^{\mp}w^\pm, \partial_{\boldsymbol{\nu}}^{\mp}w^\pm) = P^{\mp}(M_{\boldsymbol{\nu}}^\pm, \nabla_{\mathrm{T}}\cdot\boldsymbol{M}_T^\pm) = (M_{\boldsymbol{\nu}}^\pm, \nabla_{\mathrm{T}}\cdot\boldsymbol{M}_T^\pm).$$

Then,

$$\begin{aligned} \langle\!\langle \boldsymbol{M}, \boldsymbol{M}^\pm \rangle\!\rangle_{L^2(\Gamma)} &= \langle\!\langle \nabla_{\mathrm{T}}(\gamma^{\mp}w),\ \boldsymbol{M}_T^\pm \rangle\!\rangle + \langle\!\langle \partial_{\boldsymbol{\nu}}^{\mp}w,\ M_{\boldsymbol{\nu}}^\pm \rangle\!\rangle = -\langle\!\langle \gamma^{\mp}w,\ \nabla_{\mathrm{T}}\cdot\boldsymbol{M}_T^\pm \rangle\!\rangle + \langle\!\langle \partial_{\boldsymbol{\nu}}^{\mp}w,\ M_{\boldsymbol{\nu}}^\pm \rangle\!\rangle \\ &= \mp\langle\!\langle \gamma^{\mp}w,\ \partial_{\boldsymbol{\nu}}^{\mp}w^\pm \rangle\!\rangle \pm \langle\!\langle \partial_{\boldsymbol{\nu}}^{\mp}w,\ \gamma^{\mp}w^\pm \rangle\!\rangle = 0, \end{aligned}$$

where the last equality follows from the Green's identities an a density argument. Therefore $\boldsymbol{M} \perp \mathcal{M}_\pm$ as well, and since $(\mathcal{M}_\pm \oplus \mathcal{M}_0)^\perp$ is closed, the corollary follows. $\square$

4.1. **Spherical case.** In this subsection we assume that $\Gamma = \mathbb{S}^2$, the unit sphere on $\mathbb{R}^3$ and that $k > 0$. In this case, some calculations from the previous subsection can be made explicit using the Addition Theorem.

Recall that, if we let P_n^m denote the associated Legendre function of order m, then the following define a complete orthonormal system in $L^2(\mathbb{S})$ [8, Theorem 2.8] and a complete orthogonal system in $H^1(\mathbb{S})$ [23, Theorem 2.4.4]:

$$Y_n^m(x) \coloneqq \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}\ P_n^{|m|}(\cos\theta)\ e^{im\varphi} \qquad \text{for } m = -n,\dots,n, \text{ and } n = 0,1,2,\dots,$$

where $x = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. Note that $\overline{(Y_n^m(x))} = Y_n^{-m}(x)$. These functions also satisfy

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$$\Delta_{\mathrm{T}}Y_n^m = -n(n+1)\ Y_n^m. \tag{4.11}$$

Note that this implies that $\langle\!\langle Y_n^m, Y_n^m \rangle\!\rangle_{H^1(\mathbb{S})} = 1 + n(n+1)$. Given $\phi \in L^2(\mathbb{S})$ and $\boldsymbol{M} \in L^2(\mathbb{S})^3$ define the coefficients:

$$c_n^m(\phi) := \langle\!\langle Y_n^m, \phi \rangle\!\rangle_{L^2(\mathbb{S})} = \frac{\langle\!\langle Y_n^m, \phi \rangle\!\rangle_{H^1(\mathbb{S})}}{\langle\!\langle Y_n^m, Y_n^m \rangle\!\rangle_{H^1(\mathbb{S})}} \qquad \text{for } m = -n, ..., n, \text{ and } n = 0, 1, 2, ...,$$

and, for $m = -n, ..., n$ and $n = 1, 2, 3, ...,$

$$g_n^m(\boldsymbol{M}) := \frac{\langle\!\langle \nabla_{\mathrm{T}} Y_n^m, \boldsymbol{M} \rangle\!\rangle_{L^2(\mathbb{S})^3}}{n(n+1)}, \quad r_n^m(\boldsymbol{M}) := \frac{\langle\!\langle \boldsymbol{\nu} \times \nabla_{\mathrm{T}} Y_n^m, \boldsymbol{M} \rangle\!\rangle_{L^2(\mathbb{S})^3}}{n(n+1)},$$

$$g_0^0(\boldsymbol{M}) := 0 \qquad \text{and} \qquad r_0^0(\boldsymbol{M}) := 0.$$

Then, $\phi = \sum c_n^m(\phi) Y_n^m$ in $L^2(\mathbb{S})$. Note that, for any n and m, $g_n^m(\boldsymbol{M}) = g_n^m(\boldsymbol{M}_T)$ and $r_n^m(\boldsymbol{M}) = r_n^m(\boldsymbol{M}_T)$. Additionally, if $u \in H^1(\mathbb{S})$ then $u = \sum c_n^m(u) Y_n^m$ in $H^1(\mathbb{S})$ and we have:

$$g_n^m(\nabla_{\mathrm{T}} u) = c_n^m(u), \quad r_n^m(\nabla_{\mathrm{T}} u) = 0, \quad g_n^m(\boldsymbol{\nu} \times \nabla_{\mathrm{T}} u) = 0 \quad \text{and} \quad r_n^m(\boldsymbol{\nu} \times \nabla_{\mathrm{T}} u) = c_n^m(u).$$

By Hodge decomposition, there exist $u, v \in H^1(\mathbb{S})$ such that $\boldsymbol{M}_T = \nabla_{\mathrm{T}} u + \boldsymbol{\nu} \times \nabla_{\mathrm{T}} v$, and hence,

$$\begin{aligned} \boldsymbol{M}_T &= \nabla_{\mathrm{T}} \sum c_n^m(u) Y_n^m + \boldsymbol{\nu} \times \nabla_{\mathrm{T}} \sum c_n^m(v) Y_n^m \\ &= \sum g_n^m(\boldsymbol{M}) \nabla_{\mathrm{T}} Y_n^m + \sum r_n^m(\boldsymbol{M})(\boldsymbol{\nu} \times \nabla_{\mathrm{T}} Y_n^m) \end{aligned}$$

in $L^2(\mathbb{S})^3$. Therefore, $\nabla_{\mathrm{T}} \cdot \boldsymbol{M}_T = -n(n+1) \sum g_n^m(\boldsymbol{M}) Y_n^m$ in $H^{-1}(\mathbb{S})$.

Further define, for a non-negative integer n, $h_n^{(1)}$ as the spherical Hankel function of the first kind of order n, and let j_n denote the spherical Bessel function of order n. Then, since we are assuming that $k \neq 0$, we have the following Addition Theorem [8, Theorem 2.11]

$$G(x-y) = -ik \sum_{n=0}^{\infty} \sum_{m=-n}^{n} h_n^{(1)}(k|x|) Y_n^m\left(\frac{x}{|x|}\right) j_n(k|y|) \overline{\left(Y_n^m\left(\frac{y}{|y|}\right)\right)} \qquad \text{for } |x| > |y|,$$

where the series and its term by term first derivatives with respect to $|x|$ and $|y|$ are absolutely and uniformly convergent on compact subsets of $|x| > |y|$.

Then, using Fubini-Tonelli theorem we obtain:

$$SL(Y_n^m)(x) = \begin{cases} -ik\ h_n^{(1)}(k|x|)\ j_n(k)\ Y_n^m\left(\frac{x}{|x|}\right) & \text{for } |x| > 1 \\ -ik\ h_n^{(1)}(k)\ j_n(k|x|)\ Y_n^m\left(\frac{x}{|x|}\right) & \text{for } |x| < 1 \end{cases},$$

$$DL(Y_n^m)(x) = \begin{cases} -ik^2\ h_n^{(1)}(k|x|)\ j_n'(k)\ Y_n^m\left(\frac{x}{|x|}\right) & \text{for } |x| > 1 \\ -ik^2\ (h_n^{(1)})'(k)\ j_n(k|x|)\ Y_n^m\left(\frac{x}{|x|}\right) & \text{for } |x| < 1 \end{cases},$$

thus,

$$K(Y_n^m) = -\frac{1}{2} ik^2 \left(h_n^{(1)}(k) j_n'(k) + (h_n^{(1)})'(k)\ j_n(k) \right) Y_n^m \qquad T(Y_n^m) = -ik^3\ (h_n^{(1)})'(k)\ j_n'(k)\ Y_n^m$$

$$S(Y_n^m) = -ik\ h_n^{(1)}(k)\ j_n(k)\ Y_n^m \qquad K^*(Y_n^m) = K(Y_n^m),$$

Since K, T, S, K^* and $\nabla_{\mathrm{T}}\cdot : L^2(\mathbb{S})^3 \longrightarrow H^{-1}(\mathbb{S})$ are continuous, and $h_n^{(1)}(k)j_n'(k) - (h_n^{(1)})'(k)\ j_n(k) = 1/(ik^2)$, using (4.11) we get

$$P^-(M_{\boldsymbol{\nu}}, \nabla_{\mathrm{T}}\cdot \boldsymbol{M}_T) = \left(\sum_{n,m} \left[(1/2 - K(Y_n^m))c_n^m(M_{\boldsymbol{\nu}}) \ + \ S(Y_n^m)(-n(n+1)g_n^m(\boldsymbol{M}_T))\right] Y_n^m, \right.$$
$$\left. \sum_{n,m} \left[-T(Y_n^m)c_n^m(M_{\boldsymbol{\nu}}) \ + \ (1/2 + K(Y_n^m))(-n(n+1)g_n^m(\boldsymbol{M}_T))\right] Y_n^m \right)$$
$$= \left(\sum_{n,m} ikh_n^{(1)}(k)\left[kj_n'(k)c_n^m(M_{\boldsymbol{\nu}}) + j_n(k)n(n+1)g_n^m(\boldsymbol{M}_T)\right] Y_n^m, \right.$$
$$\left. \sum_{n,m} ik^2(h_n^{(1)})'(k)\left[kj_n'(k)c_n^m(M_{\boldsymbol{\nu}}) + j_n(k)n(n+1)g_n^m(\boldsymbol{M}_T)\right] Y_n^m \right),$$

and

$$P^+(M_{\boldsymbol{\nu}}, \nabla_{\mathrm{T}}\cdot \boldsymbol{M}_T) = \left(\sum_{n,m} -ikj_n(k)\left[k(h_n^{(1)})'(k)c_n^m(M_{\boldsymbol{\nu}}) + h_n^{(1)}(k)n(n+1)g_n^m(\boldsymbol{M}_T)\right] Y_n^m, \right.$$
$$\left. \sum_{n,m} -ik^2j_n'(k)\left[k(h_n^{(1)})'(k)c_n^m(M_{\boldsymbol{\nu}}) + h_n^{(1)}(k)n(n+1)g_n^m(\boldsymbol{M}_T)\right] Y_n^m \right),$$

Recall that for all n, $h_n^{(1)}(k) \neq 0 \neq (h_n^{(1)})'(k)$ for k is real and positive. Therefore,

$$\mathcal{M}_- = \{\boldsymbol{M} \in L^2(\mathbb{S})^3 : \ kj_n'(k)c_n^m(M_{\boldsymbol{\nu}}) = -j_n(k)\ n(n+1)g_n^m(\boldsymbol{M}_T) \quad \text{for } m = -n, ..., n, \text{ and } n = 0, 1, 2, ..., \}, \tag{4.12}$$

and

$$\mathcal{M}_+ = \{\boldsymbol{M} \in L^2(\mathbb{S})^3 : \ k(h_n^{(1)})'(k)\ c_n^m(M_{\boldsymbol{\nu}}) = -h_n^{(1)}(k)\ n(n+1)g_n^m(\boldsymbol{M}_T) \quad \text{for } m = -n, ..., n, \text{ and } n = 0, 1, 2, ..., \text{ such that } j_n(k) \neq 0 \text{ or } j_n'(k) \neq 0\}. \tag{4.13}$$

Since $j_0(k) = \sin(k)/k$, for no real k we get $j_0(k) = 0 = j_0'(k)$. Hence, for a $\boldsymbol{M} \in \mathcal{M}_+ + \mathcal{M}_-$, if $j_0'(k) \neq 0$, we have that $c_0^0(M_{\boldsymbol{\nu}}) = 0$. Thus, $\langle\!\langle \boldsymbol{M}, Y_0^0\boldsymbol{\nu}\rangle\!\rangle_{L^2(M)^3} = 0$. Otherwise, when $j_0'(k) = 0$ and $k > 0$, we have $P^-(Y_0^0, 0) = 0$. Therefore, using Theorem 4.2 we get the following result.

Theorem 4.4. *For a $k > 0$, if $j_0'(k) \neq 0$ (which happens a.e.), then*

$$(\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0)^\perp = \{\boldsymbol{M} \in L^2(\mathbb{S})^3 \ : \ \boldsymbol{M}_T = 0 \textit{ and } M_{\boldsymbol{\nu}} \textit{ is constant}\},$$

on the other hand, if $j_0'(k) = 0$, which happens for example when $k = 0$, then

$$L^2(\mathbb{S})^3 = \mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0.$$

App|ext

Appendix A.

A.1. **Adaptation of results from [16].** The statements in this section are either adaptations to the case $k > 0$, of directly taken from [16]. For each of them, we write in parenthesis where in [16] they can be found. For convenience, throughout this section, we will denote the function G and the operators SL, DL, S, and K by G_k, SL_k, DL_k, S_k and K_k respectively. For the operators S_k and K_k we will use the definitions given in [21] (without the $1/2$ for K_k and K_k^*) and then show that they can be extended with the required properties.

Recall $\mathfrak{C}_\alpha^\pm(x)$ from the definition of the nontangential limit. For a vector valued measurable function $\boldsymbol{\psi}$ on $\Omega_\pm$, we define the function $\mathfrak{N}_\alpha^\pm\boldsymbol{\psi}$, on Γ, such that, for $x \in \Gamma$,

$$\mathfrak{N}_\alpha^\pm\boldsymbol{\psi}(x) := \sup\{|\boldsymbol{\psi}(y)| \ : \ y \in \mathfrak{C}_\alpha^\pm(x)\},$$

taking the convention that $\mathfrak{N}_\alpha^\pm\boldsymbol{\psi}(x) = 0$ when $\mathfrak{C}_\alpha^\pm(x) = \varnothing$.

In [16, section 3.6], the Sobolev space $L^2_1(\Gamma, d\sigma)$ is defined as the subspace of $L^2(\Gamma)$ comprised of those functions φ such that $|\langle\ \varphi\ ,\ \nu_j\gamma(\partial_l f) - \nu_l\gamma(\partial_j f)\ \rangle_{L^2(\Gamma)}| \le C\|f_{|\Gamma}\|_{L^2(\Gamma)}$ for all $f \in C^1(\mathbb{R}^3)$, any $l, j \in \{1,2,3\}$ and some constant $C = C(\varphi)$, with ν_j to mean the j-th coordinate of the unit normal field on Γ. That is, if one puts as in [16] $\partial_{\tau_{l,j}} f := \nu_j\gamma(\partial_l f) - \nu_l\gamma(\partial_j f)$ for $f \in C^1(\mathbb{R}^3)$ then $\partial_{\tau_{l,j}} f$ depends only on the restriction $f_{|\Gamma}$ and members of $L^2_1(\Gamma)$ are those $\varphi \in L^2(\Gamma)$ whose distributional $\partial_{\tau_{l,j}}\varphi$ is an $L^2(\Gamma)$-function for each j, l. To justify quoting certain results from [16], we will show in the next lemma that this definition agrees with the one of the Sobolev space $H^1(\Gamma)$ made in Section 2.

lemma|tan_der **Lemma A.1.** *Given $j, l \in \{1,2,3\}$, one can define a bounded linear operator $\partial_{\tau_{i,j}} : H^1(\Gamma) \longrightarrow L^2(\Gamma)$ on letting, for any $\varphi \in H^1(\Gamma)$ and $f \in C^1(\mathbb{R}^3)$:*

$$\langle\ \partial_{\tau_{j,l}}\varphi,\ \gamma(f)\ \rangle := -\ \langle\ \varphi\ ,\ \nu_j\gamma(\partial_l f) - \nu_l\gamma(\partial_j f)\ \rangle_{L^2(\Gamma)}.$$

Moreover, a function $\varphi \in L^2(\Gamma)$ lies in $H^1(\Gamma)$ if and only if the operators $\partial_{\tau_{i,j}}$ defined above (in the weak sense) correspond to scalar product with L^2-functions.

Proof. Note that a tangent vector field on Γ can be regarded as a 1-form, defined by taking the scalar product in the tangent space at regular points. For $\{(\theta_j, U_j)\}_{j\in I}$ (I finite) a Lipschitz atlas on Γ, we say that a k-form ω is of L^2-class (here $k \in \{0,1,2\}$) if its expression in local coordinates (pullback of ω under the Lipschitz map θ_j^{-1}), say

$$(\theta_j^{-1})^*(\omega)(y) = \sum_{i_1<i_2,\cdots,<i_k} a^{\{\phi_j\}}_{i_1,\cdots,i_k}(y)\ dy_{i_1}\wedge\cdots\wedge dy_{i_k}$$

has coefficients $a^{\{\phi_j\}}_{i_1,\cdots,i_k}$ that are L^2 functions on $\theta_j(U_j)$. This notion is independent of the atlas. Now, for $f \in C_c^\infty(\mathbb{R}^3)$, it holds that

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$$(\partial_{\tau_{2,3}} f\ ,\ \partial_{\tau_{3,1}} f\ ,\ \partial_{\tau_{1,2}} f)^t = \nabla f \times \nu \tag{A.1}$$

where "$\times$" indicates the vector product and the superscript "t" means "transpose". Thus, observing that $\nu = \partial_{y_1}\theta_j^{-1} \times \partial_{y_2}\theta_j^{-1}/|\partial_{y_1}\theta_j^{-1} \times \partial_{y_2}\theta_j^{-1}|$ on $\theta_j(U_j)$, we get from the double vector product formula that the 1-form associated with $\nabla f \times \nu$ is given in local coordinates (y_1, y_2) on $\theta_j(U_j)$ by

forco
$$\Big(g_{1,1}\partial_{y_2}(f\circ\theta_j^{-1}) - g_{2,1}\partial_{y_1}(f\circ\theta_j^{-1})\Big)dy_1 + \Big(g_{1,2}\partial_{y_2}(f\circ\theta_j^{-1}) - g_{2,2}\partial_{y_1}(f\circ\theta_j^{-1})\Big)dy_2 \tag{A.2}$$

where (g_{i_1,i_2}) is the metric tensor (the Gram matrix of $\partial_{y_1}\theta_j^{-1}$, $\partial_{y_2}\theta_j^{-1}$). Since the latter is uniformly boundedly invertible on compact manifold that are local Lipschitz graphs, the fact that (A.2) is of L^2-class amounts to say that $\nabla f \circ \theta_i^{-1}$ lies in $(L^2(\theta_j(U_j)))^3$. By density of traces of $C_c^\infty(\mathbb{R}^3)$-functions in $L^2(\Gamma)$, we conclude what we want. ☐

Then, we have a lemma that was just stated on [16] since it was proven in [12]. However, we add a proof for convenience of the reader.

lemma|C_Gk **Lemma A.2** (Lemma 6.4.2)**.** *For each fixed $R > 0$ and $k > 0$, there exists a constant $C > 0$ such that, for $1 \le j \le 3$ the following estimates are uniformly satisfied for $0 < |x| < R$:*

$$|G_k(x) - G_0(x)| \le C$$
$$|\partial_j G_k(x) - \partial_j G_0(x)| \le C$$
$$|\partial_\ell\partial_j G(x) - \partial_\ell\partial_j G_0(x)|\ |x| \le C$$

Proof. Since $G_k - G_0$ is $C^\infty(\mathbb{R}^3 \smallsetminus \{0\})$, it is enough to show that the lim sup when $x \to 0$ in all of the left hand sides of the equations of the lemma are bounded by a constant depending only on k:

$$\limsup_{x\to 0} |G_k(x) - G_0(x)| = \lim_{x\to 0} \frac{\left|-1 + e^{ik|x|}\right|}{4\pi|x|} = \frac{k}{4\pi},$$

and

$$\limsup_{x\to 0}|\partial_j G_k(x)-\partial_j G_0(x)| = \limsup_{x\to 0}\left|x_j\,\frac{e^{ik|x|}k|x|+ie^{ik|x|}-i}{4\pi|x|^3}\right| \le \lim_{x\to 0}\frac{\left|x_j e^{ik|x|}k|x|+ie^{ik|x|}-i\right|}{4\pi|x|^2} = \frac{k^2}{8\pi}.$$

$$\begin{aligned}\limsup_{x\to 0}|\partial_j\partial_j G(x)-\partial_j\partial_j G_0(x)||x| &= \limsup_{x\to 0}\frac{\left|e^{ik|x|}\left(ik|x|^3-|x|^2-k^2x_j^2|x|^2-3ikx_j^2|x|+3x_j^2\right)+|x|^2-3x_j^2\right|}{4\pi|x|^4}\\ &\le \limsup_{x\to 0}\frac{\left|e^{ik|x|}\left(ik|x|^3-|x|^2\right)+|x|^2\right|}{4\pi|x|^4}+\limsup_{x\to 0}\frac{\left|e^{ik|x|}\left(-k^2x_j^2|x|^2-3ikx_j^2|x|+3x_j^2\right)-3x_j^2\right|}{4\pi|x|^4}\\ &\le \lim_{x\to 0}\frac{\left|e^{ik|x|}\left(ik|x|-1\right)+1\right|}{4\pi|x|^2}+\lim_{x\to 0}\frac{\left|e^{ik|x|}\left(-k^2|x|^2-3ik|x|+3\right)-3\right|}{4\pi|x|^2} = \frac{k^2}{4\pi},\end{aligned}$$

and, for $j\neq \ell$,

$$\begin{aligned}\limsup_{x\to 0}|\partial_\ell\partial_j G(x)-\partial_\ell\partial_j G_0(x)||x| &= \limsup_{x\to 0}\frac{\left|x_jx_\ell\left(3+e^{ik|x|}(k^2|x|^2+3ik|x|-3)\right)\right|}{4\pi|x|^4}\\ &\le \lim_{x\to 0}\frac{\left|3+e^{ik|x|}(k^2|x|^2+3ik|x|-3)\right|}{4\pi|x|^2} = \frac{k^2}{8\pi}.\end{aligned}$$

□

Then, we continue with a generalization of a relatively basic result that is just partly stated on [16] and whose proof, for the $k=0$ case, can be found as part of [5, Theorem 4.5.].

prop|limSL **Proposition A.3** (Partly stated on equation (3.6.27) and Corollary 3.6.3)**.** *Given a $\phi\in L^2(\Gamma)$, it is satisfied in the nontangential sense that $\gamma^{\pm}SL_k\phi = S_k\phi$, σ-a.e. and, for every $\alpha>0$, there exists a constant $\tilde{C}_\alpha$ such that $\|\mathfrak{N}^{\pm}_\alpha(SL_k\phi)\|_2\le \tilde{C}_\alpha\|\phi\|_2$. Also, the left equation of (3.7) is satisfied and we have the mapping property,*

q|hmt_S_k-0->1 (A.3)
$$S_k : L^2(\Gamma)\longrightarrow H^1(\Gamma).$$

Proof. Note that for any $x\in\Gamma$ and $\phi\in L^2(\Gamma)$, using the $k=0$ result,

$$\left|\int_\Gamma G_k(x-y)\phi(y)\mathrm{d}\sigma(y)\right| \le \int_\Gamma |G_k(x-y)|\;|\phi(y)|\;\mathrm{d}\sigma(y) = \int_\Gamma G_0(x-y)\;|\phi(y)|\;\mathrm{d}\sigma(y) = S_0|\phi|(x),$$

and thus, we have that in general, for σ-a.e. $x\in\Gamma$, the integral in the left equation of (3.7) defines a bounded linear operator from $L^2(\Gamma)$ to itself. Let's call this operator $\tilde{S}_k$. Now, notice the following facts; $\mathrm{Lip}(\Gamma)$ is dense in $L^2(\Gamma)$; both $\mathrm{Lip}(\Gamma)$ and $L^2(\Gamma)$ are dense in $H^{-1/2}(\Gamma)$; S_k and $\tilde{S}_k$ coincide in $\mathrm{Lip}(\Gamma)$; and the image of $\mathrm{Lip}(\Gamma)$ over $\tilde{S}_k$ belongs to $L^2(\Gamma)$. Then, S_k and $\tilde{S}_k$ must also coincide in $L^2(\Gamma)$. Thus, as a small abuse of notation we will refer to $\tilde{S}_k$ as simply S_k. Next, if $\|\phi\|_2=1$, and we take C from Lemma A.2

$$\begin{aligned}\|\nabla_{\mathrm{T}}S_k\phi-\nabla_{\mathrm{T}}S_0\phi\|_2 &= \sup_{\substack{\boldsymbol{f}\in\mathrm{Lip}_T(\Gamma)\\ \|\boldsymbol{f}\|_\infty\le 1}}\int_\Gamma\left(\int_\Gamma (G_k-G_0)(x-y)\;\phi(y)\;\mathrm{d}\sigma(y)\right)\nabla_{\mathrm{T}}\cdot\boldsymbol{f}(x)\;\mathrm{d}\sigma(x)\\ &= \sup_{\substack{\boldsymbol{f}\in\mathrm{Lip}_T(\Gamma)\\ \|\boldsymbol{f}\|_\infty\le 1}}\int_\Gamma\left(\int_\Gamma (\nabla G_k-\nabla G_0)(x-y)\cdot\boldsymbol{f}(x)\;\mathrm{d}\sigma(x)\right)\phi(y)\;\mathrm{d}\sigma(y)\\ &= \int_\Gamma\left(\int_\Gamma |(\nabla G_k-\nabla G_0)(x-y)|\;\mathrm{d}\sigma(x)\right)|\phi(y)|\;\mathrm{d}\sigma(y)\\ &\le \sqrt{3}C\sigma(\Gamma)\|\phi\|_1 \le \sqrt{3}C\sigma(\Gamma)(\|\phi\|_2^2+\sigma(\Gamma)) = \sqrt{3}C\sigma(\Gamma)(1+\sigma(\Gamma)),\end{aligned}$$

Then, as $S_0 : L^2(\Gamma) \longrightarrow H^1(\Gamma)$ is bounded and $\|S_k\phi\|_{H^1(\Gamma)} \le \|S_k\phi - S_0\phi\|_{H^1(\Gamma)} + \|S_0\phi\|_{H^1(\Gamma)}$, we obtain that S_k is also a bounded linear operator from $L^2(\Gamma)$ to $H^1(\Gamma)$.

Take $\alpha > 0$, $x \in \Gamma$ and $y \in \mathfrak{C}^{\pm}_{\alpha}(x)$. Then for any $z \in \Gamma$

$$|y - z| \ge \operatorname{dist}(y, \Gamma) \ge \frac{|x-y|}{\alpha+1} \quad \text{so,} \quad |y-z|(\alpha+2) \ge |x-y| + |y-z| \ge |x-z|. \tag{A.4}$$

Thus,

$$|SL_k\phi(y)| = \left|\int_\Gamma G_k(y-z)\phi(z)\mathrm{d}\sigma(z)\right| \le \int_\Gamma \frac{(\alpha+2)|\phi(z)|}{4\pi|x-z|}\mathrm{d}\sigma(z) = (\alpha+2)S_0|\phi|(x).$$

Hence, we can use Dominated convergence and the result for $k = 0$, to obtain for σ-a.e. $x \in \Gamma$, that it is satisfied in the nontangential sense $\gamma^{\pm}SL_k\phi = S_k\phi$. Also, taking $\tilde{C}_\alpha$ to be the operator norm of S_0 times $\alpha + 2$ we obtain that $\|\mathfrak{N}^{\pm}_\alpha(SL_k\phi)\|_2 \le \tilde{C}_\alpha\|\phi\|_2$. ☐

prop|K_k **Proposition A.4** (Proposition 3.3.2). *Take a $\phi \in L^2(\Gamma)$. For $f = \phi$, the principal value of equation (3.7) exists for σ-a.e. $x \in \Gamma$ and it can be used to extend the operator K_k to*

$$K_k : L^2(\Gamma) \longrightarrow L^2(\Gamma)$$

which is bounded. Furthermore, the right equation of (3.8) is satisfied in the nontangential limit sense and for every $\alpha > 0$, we have that $\|\mathfrak{N}^{\pm}_\alpha(DL_k\phi)\|_2 \le \tilde{C}_\alpha\|\phi\|_2$ for some $\tilde{C}_\alpha > 0$ depending only on Γ, k and α.

Proof. By [16, Proposition 3.3.2] the result is valid for $k = 0$. Take any $\phi \in L^2(\Gamma)$ and $x \in \Gamma$ such that $K_0\phi(x)$ is well-defined, which is σ-a.e. Define for any $\varepsilon > 0$

$$K_k^\varepsilon\phi(x) := \int\limits_{\substack{y\in\Gamma\\ |x-y|>\varepsilon}} \partial_{\boldsymbol{\nu},y}G_k(x-y)\ \phi(y)\ \mathrm{d}\sigma(y) = -\int\limits_{\substack{y\in\Gamma\\ |x-y|>\varepsilon}} (\nabla G_k)(x-y)\cdot\boldsymbol{\nu}(y)\ \phi(y)\ \mathrm{d}\sigma(y).$$

Then, K_0^ε defines a bounded linear operator from $L^2(\Gamma)$ to itself and, whenever $K_0\phi(x)$ is well defined, $K_0^\varepsilon\phi(x) \to K_0\phi(x)$ as $\varepsilon \to 0$. Thus, for any sequence $(\varepsilon_n)_n$ such that $\varepsilon_n \to 0$ as $n \to \infty$, the sequence $(K_0^{\varepsilon_n}\phi(x))_n$ is Cauchy, whenever $K_0\phi(x)$ is well-defined. Hence, showing that the principal value of equation (3.7) exists for x is equivalent to showing that the sequence $(K_k^{\varepsilon_n}\phi(x))_n$ is Cauchy as well. Take $m > n$ and see that,

$$\begin{aligned}
|K_k^{\varepsilon_n}\phi(x) - K_k^{\varepsilon_m}\phi(x)| &\le |K_k^{\varepsilon_n}\phi(x) - K_k^{\varepsilon_m}\phi(x) - K_0^{\varepsilon_n}\phi(x) + K_0^{\varepsilon_m}\phi(x)| \ + \ |K_0^{\varepsilon_n}\phi(x) - K_0^{\varepsilon_m}\phi(x)| \\
&\le \int\limits_{\substack{y\in\Gamma\\ \varepsilon_n > |x-y| > \varepsilon_m}} |(\nabla G_k - \nabla G_0)(x-y)|\ |\phi(y)|\ \mathrm{d}\sigma(y) \ + \ |K_0^{\varepsilon_n}\phi(x) - K_0^{\varepsilon_m}\phi(x)| \\
&\le \int\limits_{\substack{y\in\Gamma\\ \varepsilon_n > |x-y| > \varepsilon_m}} \sqrt{3}C\ |\phi(y)|\ \mathrm{d}\sigma(y) \ + \ |K_0^{\varepsilon_n}\phi(x) - K_0^{\varepsilon_m}\phi(x)|
\end{aligned}$$

where the constant C, taken from Lemma A.2, depends only on k and the size of the bounded set Γ. Thus, the integrability of ϕ and the fact that $(K_0^{\varepsilon_n}\phi(x))_{n=1}^\infty$ is Cauchy imply that $(K_k^{\varepsilon_n}\phi(x))_{n=1}^\infty$ is Cauchy as well. Since the result is valid for $k = 0$, the value $K_0\phi(x)$ is well defined for σ-a.e. $x \in \Gamma$. Then, $\tilde{K}_k\phi(x) := \lim_{\varepsilon\to 0} K_k^\varepsilon\phi(x)$ is also well defined for σ-a.e. $x \in \Gamma$, and it defines a measurable function since it is the point-wise limit of L^2 functions.

Note that $\tilde{K}_k$ defines a linear operator on $L^2(\Gamma)$. Take now any $\phi \in L^2(\Gamma)$ with $\|\phi\|_2 = 1$. Then using, Fatou's lemma we get

$$\begin{aligned}
\left\|\tilde{K}_k\phi - K_0\phi\right\|_2^2 &\leq \liminf_{\varepsilon\to 0} \int_{x\in\Gamma} \left| \int_{\substack{y\,\in\,\Gamma\\ |x-y|\,>\,\varepsilon}} (\nabla G_k - \nabla G_0)(x-y)\cdot \boldsymbol{\nu}(y)\ \phi(y)\ \mathrm{d}\sigma(y) \right|^2 \mathrm{d}\sigma(x) \\
&\leq \liminf_{\varepsilon\to 0} \int_{x\in\Gamma} \left(\int_{\substack{y\,\in\,\Gamma\\ |x-y|\,>\,\varepsilon}} |(\nabla G_k - \nabla G_0)(x-y)|\ |\phi(y)|\ \mathrm{d}\sigma(y) \right)^2 \mathrm{d}\sigma(x) \\
&\leq 3C^2\sigma(\Gamma)\|\phi\|_1^2 \leq 3C^2\sigma(\Gamma)(\|\phi\|_2^2 + \sigma(\Gamma))^2 = 3C^2\sigma(\Gamma)(1+\sigma(\Gamma))^2,
\end{aligned}$$

with the same constant C as before. Then, as K_0 is bounded and $\|\tilde{K}_k\phi\|_2 \leq \|\tilde{K}_k\phi - K_0\phi\|_2 + \|K_0\phi\|_2$, we obtain that $\tilde{K}_k$ is bounded having $L^2(\Gamma)$ as its image. Now, using an argument analogous to the one in Lemma A.3 for $\tilde{S}_k$, we can show that $\tilde{K}_k$ coincides with K_k in $L^2(\Gamma)$ and thus, as a small abuse of notation we will refer to $\tilde{K}_k$ as just K_k.

Fix a $\phi \in L^2(\Gamma)$ such that $\|\phi\|_2 = 1$. By [16, equation (3.3.6)], for any $\alpha > 0$ there exists a constant C_α such that

$$\|\mathfrak{N}_\alpha^\pm(DL_0\phi)\|_2 \leq C_\alpha.$$

On the other hand,

$$\begin{aligned}
|\mathfrak{N}_\alpha^\pm(DL_k\phi)(x) - \mathfrak{N}_\alpha^\pm(DL_0\phi)(x)| &\leq \mathfrak{N}_\alpha^\pm(DL_k\phi - DL_0\phi)(x) \\
&= \sup_{z\in\mathfrak{C}_\alpha^\pm(x)} \left| \int_\Gamma (\nabla G_k - \nabla G_0)(z-y)\cdot\boldsymbol{\nu}(y)\ \phi(y)\ \mathrm{d}\sigma(y) \right| \\
&\leq \sup_{z\in\mathfrak{C}_\alpha^\pm(x)} \int_\Gamma |(\nabla G_k - \nabla G_0)(z-y)|\ |\phi(y)|\ \mathrm{d}\sigma(y) \\
&\leq \sqrt{3}C\|\phi\|_1 \leq \sqrt{3}C(1+\sigma(\Gamma)).
\end{aligned}$$

Then,

$$\begin{aligned}
\|\mathfrak{N}_\alpha^\pm(DL_k\phi)\|_2 &\leq \|\mathfrak{N}_\alpha^\pm(DL_k\phi) - \mathfrak{N}_\alpha^\pm(DL_0\phi)\|_2 + \|\mathfrak{N}_\alpha^\pm(DL_0\phi)\|_2 \\
&\leq \sqrt{3\sigma(\Gamma)}C(1+\sigma(\Gamma)) + C_\alpha =: \tilde{C}_\alpha.
\end{aligned}$$

Therefore, for a general $\phi \in L^2(\Gamma)$ we get

eq|DLmaxnontan

$$\|\mathfrak{N}_\alpha^\pm(DL_k\phi)\|_2 \leq \tilde{C}_\alpha\|\phi\|_2. \tag{A.5}$$

With a slightly modified argument to the one of the proof of [16, Proposition 3.3.2], it follows that for all $f \in \mathrm{Lip}(\Gamma)$, the nontangential limit $\gamma^\pm DL_k f$ exists and satisfies the right equation of (3.8).

We will prove that the nontangential limits $\gamma^\pm DL\phi(x)$ exists for σ-a.e. $x \in \Gamma$ for real valued functions ϕ but the result for the complex valued ones follows immediately by linearity. Take now any real-valued $\phi \in L^2(\Gamma)$ and, using the the density of $\mathrm{Lip}(\Gamma)$ in $L^2(\Gamma)$, we can take a sequence $(f_n)_n \subset \mathrm{Lip}(\Gamma)$ of real value functions that converge to ϕ in $L^2(\Gamma)$. Then define, for any real-valued measurable function ψ on $\Omega_\pm$ and for any $x \in \Gamma$ such that $x \in \overline{\mathfrak{C}_\alpha^\pm(x)}$ (which by [16, Proposition 3.3.1], happens for σ-a.e. $x \in \Gamma$),

eq|trace_infsup

$$\gamma_{\alpha,\inf}^\pm\psi(x) := \liminf_{\substack{y\,\to\,x\\ y\,\in\,\mathfrak{C}_\alpha^\pm(x)}} \psi(x) \quad \text{and} \quad \gamma_{\alpha,\sup}^\pm\psi(x) := \limsup_{\substack{y\,\to\,x\\ y\,\in\,\mathfrak{C}_\alpha^\pm(x)}} \psi(x), \tag{A.6}$$

and denote the resulting function on Γ by $\gamma^{\pm}_{\alpha,\inf}\psi$ and $\gamma^{\pm}_{\alpha,\sup}\psi$, respectively. Then, using Equation (A.5)

$$\begin{aligned}\|\gamma^{\pm}_{\alpha,\inf} DL_k\phi - \gamma^{\pm} DL_k f_n\|_2 &= \|\gamma^{\pm}_{\alpha,\inf} DL_k\phi - \gamma^{\pm}_{\alpha,\sup} DL_k f_n\|_2 \le \|\gamma^{\pm}_{\alpha,\inf} DL_k(\phi - f_n)\|_2 \\ &\le \|\mathfrak{N}^{\pm}_{\alpha} DL_k(\phi - f_n)\|_2 \le \tilde{C}_\alpha \|(\phi - f_n)\|_2\end{aligned}$$

and

$$\begin{aligned}\|\gamma^{\pm}_{\alpha,\sup} DL_k\phi - \gamma^{\pm} DL_k f_n\|_2 &= \|\gamma^{\pm}_{\alpha,\sup} DL_k\phi - \gamma^{\pm}_{\alpha,\sup} DL_k f_n\|_2 \le \|\gamma^{\pm}_{\alpha,\sup} DL_k(\phi - f_n)\|_2 \\ &\le \|\mathfrak{N}^{\pm}_{\alpha} DL_k(\phi - f_n)\|_2 \le \tilde{C}_\alpha \|(\phi - f_n)\|_2.\end{aligned}$$

This implies, by the convergence of $(f_n)_n$ to ϕ in $L^2(\Gamma)$, that for any $\alpha > 0$ it is satisfied that $\gamma^{\pm}_{\alpha,\inf}\psi(x) = \gamma^{\pm}_{\alpha,\sup}\psi(x)$ for σ-a.e. $x \in \Gamma$. Hence, for any $\alpha > 0$ the limit $\gamma^{\pm}_{\alpha} DL\phi(x)$ exists for σ-a.e. $x \in \Gamma$. Next, note that for any $x \in \Gamma$ and $\alpha > \beta > 0$, if $\gamma^{\pm}_{\alpha} DL\phi(x)$ exists then $\gamma^{\pm}_{\beta} DL\phi(x)$ also exists and is equal to $\gamma^{\pm}_{\alpha} DL\phi(x)$. Thus, by taking a sequence of $\alpha_n \to \infty$, we obtain that for σ-a.e. $x \in \Gamma$, the nontangential limit $\gamma^{\pm} DL\phi(x)$ exists.

Finally, by Remark 1, the nontangential limit $\gamma^{\pm} DL\phi(x)$ is equal to the classical trace and therefore, by the density of $\mathrm{Lip}(\Gamma)$ in $L^2(\Gamma)$, the continuity of operator $K_k : L^2(\Gamma) \longrightarrow L^2(\Gamma)$ and $K_k : H^{1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma)$, we obtain that $\gamma^{\pm} DL\phi(x)$ satisfies the right equation of (3.8) in the nontangential sense. □

prop|gradDL

Proposition A.5 (Proposition 3.6.2)**.** *For each $\varphi \in H^1(\Gamma)$, the nontangential limit $\gamma^{\pm}\partial_j DL_k\varphi$ exists σ-a.e. on Γ, for each $j = 1, 2, 3$. Also, $\tilde{C}_\alpha > 0$ can be taken such that,*

eq|Na_gradDL

$$\|\mathfrak{N}^{\pm}_{\alpha}(\nabla DL_k\varphi)\|_2 \le \tilde{C}_\alpha \|\varphi\|_{H^1(\Gamma)}. \tag{A.7}$$

Finally, the restriction of K_k to $H^1(\Gamma)$ is bounded as an operator on $H^1(\Gamma)$ and we get the mapping property,

$$K_k : H^1(\Gamma) \longrightarrow H^1(\Gamma).$$

Proof. Adapting the proof of [16, Proposition 3.6.2], take any $x \in \Omega_{\pm}$ and $j = 1, 2, 3$. Then,

derexp

$$\begin{aligned}\partial_j DL_k\varphi(x) &= -\int_\Gamma \sum_{l=1}^{3} [\partial_j\partial_l G_k](x-y)\nu_l(y)\varphi(y)\mathrm{d}\sigma(y) \\ &= \int_\Gamma \varphi(y)\left(k^2 G_k(x-y)\nu_j(y) + \sum_{l\neq j}[\partial_l\partial_l G_k](x-y)\nu_j(y) - [\partial_j\partial_l G_k](x-y)\nu_l(y)\right)\mathrm{d}\sigma(y) \\ &= k^2 SL_k(\varphi\nu_j) + \sum_{l\neq j}\int_\Gamma \partial_{\tau_{j,l}}\varphi(y)\,\partial_l G_k(x-y)\,\mathrm{d}\sigma(y), \end{aligned} \tag{A.8}$$

where the second inequality uses the fact that $\Delta G + k^2 G = 0$ on $\mathbb{R}^3 \setminus \{0\}$ and the third uses Lemma A.1. The first term in (A.8) is only weakly singular and can be handled as in Lemma A.3. As for the second term, recalling that the result is known for the case $k = 0$ [27, Lemma 5.7], we are left to prove: (i) the existence of the nontangential limit a.e. on Γ and (ii) the domination of the L^2-norm of the nontangential maximal function by $C\|\varphi\|_{H^1(\Gamma)}$, this time for the quantity

$$\sum_{l\neq j}\int_\Gamma \partial_{\tau_{j,l}}\varphi(y)\left(\partial_l G_k(x-y) - \partial_l G_0(x-y)\right)\mathrm{d}\sigma(y).$$

Now, both (i) and (ii) follow by dominated convergence from the second inequality in Lemma A.2. □

prop|sl-1

Proposition A.6 (Proposition 3.6.4)**.** *The operator $S_k : H^{-1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma)$ can be extended to the bounded linear operator*

$$S_k : H^{-1}(\Gamma) \longrightarrow L^2(\Gamma),$$

which is the dual of $(S_k)_{|H^1(\Gamma)}$. Also, it is satisfied in the nontangential sense that

$$\gamma^{\pm} SL_k\psi = S_k\psi, \quad \sigma\text{-a.e. for every } \psi \in H^{-1}(\Gamma)$$

and, for every $\alpha > 0$, there exists a constant $\tilde{C}_\alpha$ such that,

$$\|\mathfrak{N}_\alpha^{\pm}(SL_k\psi)\|_2 \le \tilde{C}_\alpha \|\psi\|_{H^{-1}(\Gamma)}.$$

Proof. Note that by Lemma A.3 and Equation (3.7), for every $\phi \in L^2(\Gamma)$, we get $(S_k)^*_{|H^1(\Gamma)}(\phi) = S_k(\phi)$. Then, using again density of $H^s(\Gamma)$ in $H^t(\Gamma)$ for $t < s$, and Lemma A.3, we obtain that $(S_k)^*_{|H^1(\Gamma)}$ is indeed an extension of $S_k : H^{-1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma)$. The rest of the proof follows from similar arguments to Proposition A.4. □

ace_partial_nu **Proposition A.7** (Proposition 6.3.1). *For any $\phi \in L^2(\Gamma)$ we get*

$$\begin{aligned}\partial_{\boldsymbol{\nu}}^{\pm} SL_k\phi &= \left(\mp\frac{1}{2}Id + K_k^*\right)\phi\\ &= \boldsymbol{\nu}\cdot\gamma^{\pm}(\nabla SL_k\phi)\end{aligned}$$

Proof. The first equality is just the classical result [21, Equation (7.5)]. For the second equality, it can be shown, similarly as in the previous lemmas, that $\gamma^{\pm}\circ\partial_j SL_k - \gamma^{\pm}\circ\partial_j SL_0$ defines a bounded linear operator from $L^2(\partial)$ to itself, so that, by [16, Proposition 6.3.1], $\gamma^{\pm}\circ\partial_j SL_k$ is as well bounded Finally, we can show the result for Lipschitz functions, dividing the integral as in the proof of [16, Proposition 3.3.2] and also integrating against a test function; and finish the proof by a density argument. □

A.2. **Auxiliary regularity results.** In this section, we state an prove a couple of lemmas which are folklore but not easy to find in the literature.

lemma|reg_L **Lemma A.8.** *For $\Omega_+ \subset \mathbb{R}^3$ a bounded Lipschitz domain, the map $S_0 : L^2(\Gamma) \to H^1(\Gamma)$ is an isomorphism. Moreover, for each $f \in L^2(\Gamma)$, the harmonic function $SL_0 f$ has gradient with nontangential maximal function $\mathfrak{N}_\alpha^{\pm}(|\nabla SL_0 f|) \in L^2(\Gamma)$. In addition, $SL_0 f$ lies in $H^{3/2}(\Omega)$.*

Proof. We adopt the notation of Lemma A.12: $\Gamma_1, ..., \Gamma_l$ are the components of Γ ordered so that the connected components $O_1, ..., O_l$ of $\mathbb{R}^3 \setminus \overline{\Omega}$ satisfy $O_1 = \mathrm{Ext}(\Gamma_1)$ and $O_j = \mathrm{Int}(\Gamma_j)$ for $j \neq 1$. When $l = 1$, the lemma follows from [27, Theorem 3.3 & Corollary 3.5], except for the last statement. The latter is made in [19, Remark (b)], but that part of the argument based on interpolation which is given there is wrong. Instead, one can observe like these authors that $x \mapsto |\partial_i\partial_j SL_0 f(x)|\mathrm{dist}(x,\Gamma)^{1/2} \in L^2(\Omega)$ for $1 \le i, j \le n$ (this follows from [10, Theorem 1] using Fubini's theorem), and appeal to [18, Theorem 4.1] to obtain that $SL_0 f \in H^{3/2}(\Omega)$. In the general case, let us write $S_0(f_j)$ (resp. $SL_0(f_j)$) for the single layer potential of $f_j \in L^2(\Gamma_j)$ on Γ_j (resp. on $\mathbb{R}^3 \setminus \Gamma_j$), and consider the map $F : \Pi_j L^2(\Gamma_j) \to \Pi_j H^1(\Gamma_j)$ given by

$$F(f_1, \cdots, f_l) := \left(S_0(f_j) + \sum_{k\neq j}\gamma_{\Gamma_j} SL_0(f_k)\right)_{j=1}^{l}.$$

Clearly, by the case $l = 1$, this map is of the form $J+K$ where $J(f_1, \cdots, f_l) = (S_0(f_j))_{j=1}^l$ is invertible and K is a compact operator. Moreover F is injective, for if $F(f_1, \cdots, f_l) = 0$ then the harmonic function $\sum_j SL_0(f_j)$ is identically zero in $\Omega_\pm$ as it has vanishing nontangential limit a.e on Γ and $L^2(\Gamma)$-nontangential maximal function by the case $l = 1$ and the smoothness of $SL_0 f_j$ across Γ_k for $k \neq j$, so that we can apply [9, Theorems 1 & 3] (note that $\sum_j SL_0(f_j)$ is zero at infinity by construction); taking the Laplacian, we conclude that all f_j are zero, thereby proving the announced injectivity. Thus, by a well-known theorem of F. Riesz, F is an isomorphism, and since $S_0 f = \sum_j S_0(f_j)$ when we put $f_j = f_{|\Gamma_j}$ the fact that $\mathfrak{N}_\alpha|\nabla SL_0(f)|$ lies in $L^2(\Gamma)$ and that $SL_0 f \in H^{3/2}(\Omega)$ now follows immediately from the case $l = 1$. □

lemma|reg_J **Lemma A.9.** *Let $\Omega_+ \subset \mathbb{R}^3$ be a bounded Lipschitz domain, $(\phi, \psi) \in L^2(\Gamma) \times H^{-1}(\Gamma)$ and $u = \widetilde{\mathcal{F}}(\phi, \psi)$. If $\gamma^+ u \in H^1(\Gamma)$ then $u \in H^{3/2}(\Omega_+)$, and if $\gamma^- u \in H^1(\Gamma)$ then $u \in H^{3/2}_\ell(\Omega_-)$.*

Proof. We only prove the statement for $\gamma^- u$, as the case of $\gamma^+ u$ is analogous but simpler. Let $\mathbb{B} \subset \mathbb{R}^3$ be an open ball centered at 0 containing $\overline{\Omega_+}$, and let $u' = u_{|\mathbb{B}\setminus\overline{\Omega_+}}$ which is square integrable by remark 3.

By [18, Theorem B], there is a $w \in H^{3/2}(\mathbb{B} \setminus \overline{\Omega_+})$ such that $\Delta w = -k^2 u'$ and $\gamma_{\mathbb{B}\setminus\overline{\Omega_+}} w = 0$. Note that $\gamma_{\mathbb{B}\setminus\overline{\Omega_+}} u' \in H^1(\partial(\mathbb{B} \setminus \overline{\Omega_+}))$, since $\gamma^- u \in H^1(\Gamma)$ by assumption and u is analytic on Ω_-. So, by Lemma A.8, there is a harmonic function $v \in H^{3/2}(\mathbb{B} \setminus \overline{\Omega_+})$ whose gradient has $\mathfrak{N}_\alpha |\nabla SL_0(f)| \in L^2(\Gamma)$, and whose nontangential limit a.e. on Γ is $\gamma_{\mathbb{B}\setminus\overline{\Omega_+}} u'$. Hence, as $v + w \in H^{3/2}(\mathbb{B} \setminus \overline{\Omega_+})$, it is enough to show that $h := u' - v - w$ is the zero fonction. For this we shall prove that it lies in $H^1(\mathbb{B} \setminus \overline{\Omega_+})$ and has zero trace; since it is harmonic by construction, this will achieve the proof. Now, $h \in H^1(\mathbb{B} \setminus \overline{\Omega_+})$ if and only if u' does, and remark 3 together with the third inequality in Lemma A.2 entail in view of Lemma A.8 that u' is the sum of a harmonic function of the form $SL_0 f$ with $f \in L^2(\Gamma)$ (that lies in $H^1(\mathbb{B} \setminus \overline{\Omega_+})$ plus a function with nontangentially bounded derivative (because $x \mapsto 1/|x|$ is locally integrable in dimension 2). Altogether, $u' \in H^1(\mathbb{B} \setminus \overline{\Omega_+})$, and $h \in H^1(\mathbb{B} \setminus \overline{\Omega_+})$ as well. Finally, the trace of w is zero and the nontangential limit of v, which is also its trace, is $\gamma_{\mathbb{B}\setminus\overline{\Omega_+}} u'$. Hence h has zero trace, as wanted. ☐

ma|inv_LapBel

Lemma A.10. *Let $\Gamma \subset \mathbb{R}^3$ be the boundary of a bounded Lipschitz domain and let $\{\Gamma_j\}_{j\in J}$ be its connected components. If $\psi \in H^{-1}(\Gamma)$ is such that for every $j \in J$, $\langle \psi, 1_{\Gamma_j} \rangle = 0$, then there exists a $\varphi_\psi \in H^1(\Gamma)$ such that $\Delta_{\mathrm{T}} \varphi_\psi = \psi$.*

Proof. Let Z denote the space $\{\varphi \in H^1(\Gamma) \,:\, \text{for every } j \in J,\ \langle \varphi, 1_{\Gamma_j} \rangle = 0\}$ together with the inner product $\langle\!\langle \varphi, \tilde\varphi \rangle\!\rangle_Z := \langle\!\langle \nabla_{\mathrm{T}} \varphi, \nabla_{\mathrm{T}} \tilde\varphi \rangle\!\rangle_{L^2(\Gamma)^3}$. By the Poincaré inequality (obtained from its Euclidean version applied in a minimal system of finitely many charts (V_j, Φ_j) with Lipschitz smooth image that cover Γ to bound $\|\varphi - \int_{V_j \setminus (\cup_{k\neq j} V_k)} \varphi\|_{L^2(V_j)}$ by $K_j \|\nabla \varphi\|_{L^2(V_j)}$ for each j), one checks that Z is a Hilbert space. Pick $\psi \in H^{-1}(\Gamma)$ such that, for every $j \in J$, $\langle \psi, 1_{\Gamma_j} \rangle = 0$. Using the Poincaré inequality again, the function $\varphi \mapsto -\langle \psi, \varphi \rangle$ belongs to the dual of Z. Thus there exists a $\varphi_\psi \in Z$ such that, for every $\varphi \in Z$, $\langle \psi, \varphi \rangle = -\langle\!\langle \overline{\varphi_\psi}, \varphi \rangle\!\rangle_Z$. Take now any $\varphi \in H^1(\Gamma)$ and let, for any $j \in J$, $\alpha_j = \sigma(\Gamma_j)^{-1} \langle \varphi, 1_{\Gamma_j} \rangle$. Then,

$$\begin{aligned}\langle \psi, \varphi \rangle &= \left\langle \psi \ ,\ \varphi - \sum_{j\in J} \alpha_j 1_{\Gamma_j} + \sum_{j\in J} \alpha_j 1_{\Gamma_j} \right\rangle = \left\langle \psi \ ,\ \varphi - \sum_{j\in J} \alpha_j 1_{\Gamma_j} \right\rangle = -\left\langle\!\!\left\langle \overline{\varphi_\psi} \ ,\ \varphi - \sum_{j\in J} \alpha_j 1_{\Gamma_j} \right\rangle\!\!\right\rangle_Z \\ &= -\left\langle \nabla_{\mathrm{T}} \varphi_\psi \ ,\ \nabla_{\mathrm{T}} \left(\varphi - \sum_{j\in J} \alpha_j 1_{\Gamma_j} \right) \right\rangle_{L^2(\Gamma)^3} = -\langle \nabla_{\mathrm{T}} \varphi_\psi, \nabla_{\mathrm{T}} \varphi \rangle_{L^2(\Gamma)^3} = \langle \Delta_{\mathrm{T}} \varphi_\psi, \varphi \rangle,\end{aligned}$$

and hence $\Delta_{\mathrm{T}} \varphi_\psi = \psi$. ☐

A.3. **Basic topological facts.** Using the fact that all surfaces embedded in $\mathbb{R}^3$ are triangulable [20, Theorem 5.12], the following lemma can be found in [22, Corollary 74.2]. This is generally true for any connected compact hypersurface on $\mathbb{R}^n$ and follows as a consequence of *Alexander duality* [15, Corollary 3.45], but the proof is more involved.

emma|Alexander

Lemma A.11. *Take a connected surface $\Gamma \subset \mathbb{R}^3$ which is compact as a topological space.*

Then the set $\mathbb{R}^3 \setminus \Gamma$ has two connected components; one bounded, which we will denote by $\mathrm{Int}(\Gamma)$*, and another unbounded, which we will denote by* $\mathrm{Ext}(\Gamma)$.

Furthermore, $\partial(\mathrm{Int}(\Gamma)) = \Gamma = \partial(\mathrm{Ext}(\Gamma))$.

We say that a set $\Gamma \subset \mathbb{R}^3$ is *locally a Lipschitz graph* if for every $x \in \Gamma$ there exists an open ball $\mathbb{B} \subset \mathbb{R}^3$, a $h > 0$, a plane $H \subset \mathbb{R}^3$ passing through s and with a normal unit vector $\boldsymbol{\nu}$, and a real-valued Lipschitz continuous function g on H such that the set defined as

$$C := \{x + t\boldsymbol{\nu} \ :\ x \in \mathbb{B} \cap H,\ -h < t < h\},$$

satisfies:

$$C \cap \Gamma = \{x + t\boldsymbol{\nu} \ :\ x \in \mathbb{B} \cap H,\ t = g(x)\}.$$

ect_lip_bound

Lemma A.12. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then, Γ has finitely many connected components, say $\Gamma_1, \ldots, \Gamma_l$, each of which is locally a Lipschitz graph in $\mathbb{R}^3$.*

Moreover, the connected components of $\mathbb{R}^3 \setminus \overline{\Omega}$ consist of l Lipschitz domains $O_1, \ldots, O_l$, and with a suitable ordering $O_1 = \mathrm{Ext}(\Gamma_1)$ while $O_j = \mathrm{Int}(\Gamma_j)$ for $j \neq 1$.

Proof. The connected components $\Omega \subset \mathbb{R}^3$ are finite in number; otherwise indeed, there would exist a sequence $(\Omega_k)_k$ of such components, with $\Omega_k \cap \Omega_j = \varnothing$ for $k \neq j$. Then, we could construct a sequence $(x_k)_k \in \Omega_k$ such that x_k remains at bounded distance from $\Gamma_k \subset \Gamma$, hence x_k would be bounded and extracting a subsequence if necessary we might assume that x_k converges in $\mathbb{R}^3$ to some y. However, this is impossible for y cannot lie in Ω since the connected components of the latter are open, nor can it lie in $\mathbb{R}^3 \setminus \overline{\Omega}$, and it cannot belong to Γ either because, as Γ is a compact Lipschitz manifold which is locally a Lipschitz graph, each $x \in \Gamma$ has a neighborhood whose intersections with both Ω and Γ are connected. Consequently, by compactness, Γ has finitely many connected components, say $\Gamma_1, ..., \Gamma_l$, and each Γ_j is locally a Lipschitz graph in $\mathbb{R}^3$.

As Ω is connected by assumption, for each $j \in \{1, ..., l\}$ one of the following is true; either $\Omega \subset \text{Int}(\Gamma_j)$, so that $\overline{\Omega} \subset \overline{\text{Int}(\Gamma_j)}$ and then, using Lemma A.11, $\text{Ext}(\Gamma_j) \subset \mathbb{R}^3 \setminus \overline{\Omega}$; or else $\Omega \subset \text{Ext}(\Gamma_j)$ and then, analogously, $\text{Int}(\Gamma_j) \subset \mathbb{R}^3 \setminus \overline{\Omega}$. Since there is exactly one unbounded connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$, say O_1, it must contain $\text{Ext}(\Gamma_j)$ for all j such that $\Omega \subset \text{Int}(\Gamma_j)$; let us enumerate these j as $j_1, ..., j_m$. For $1 \leq i, k \leq m$, it holds that $\text{Int}(\Gamma_{j_i}) \cap \text{Int}(\Gamma_{j_k}) \neq \varnothing$ because Ω lies in this intersection, and since the Γ_j are disjoint one of these interiors is included in the other, say $\text{Int}(\Gamma_{j_i}) \subset \text{Int}(\Gamma_{j_k})$. But if $j_i \neq j_k$, then $\Gamma_{j_k} \subset \text{Ext}(\Gamma_{j_i})$ and the latter is contained in O_1, a contradiction. Consequently, $m = 1$ and Ω lies interior to exactly one of the Γ_j, say Γ_1. Necessarily then, $O_1 = \text{Ext}(\Gamma_1)$ because O_1 cannot strictly contain $\text{Ext}(\Gamma_1)$ without containing a point of Γ_1, which is impossible. Likewise, $\Omega \subset \text{Ext}(\Gamma_j)$ for $j \neq 1$ and then $\text{Int}(\Gamma_j)$ is a connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$. Next, the closure of every bounded connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$ must meet some Γ_j, and necessarily $j \neq 1$ for each point of Γ_1 has a neighborhood included in $\overline{O_1} \cup \Omega$, by the local Lipschitz graph property. Hence, this connected component meets $\text{Int}(\Gamma_j)$ for some $j \neq 1$, therefore it must coincide with $\text{Int}(\Gamma_j)$. Finally, due to Lemma A.11 and the definition of locally Lipschitz graphs, for each $j \in \{1, ..., l\}$ both $\text{Int}(\Gamma_j)$ and $\text{Ext}(\Gamma_j)$ are Lipschitz domains. ☐

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